

Janina Klapyta

On Some Partial Differential-Functional Equations

The present paper deals with the question of the global existence of solutions of some partial differential-functional equations of the n th order, of the hyperbolic type, with certain initial-boundary conditions of Darboux type.

The method used in this paper is a modification of the method first introduced by L. Tonelli ([2], [3]) for ordinary differential equations. This method has been also used in [1]. The results presented here are some generalizations of results obtained by A. Pelczar ([1]) for systems of integral functional equations of the type

$$y(x) = \psi(x) \quad \text{for } x \in A^* (A^* \text{ is defined below}),$$

$$y(x) = \lambda(x) + \int_a^x f(t, E(t, y)) dt \quad \text{for } x \in [a, c].$$

In the present paper we consider more general equations; the right-hand sides of these depend also on derivatives of unknown functions.

In the case when the set J of multiindices (see definitions below) is empty, the results given here reduce to the result of A. Pelczar ([1]). If $J = \emptyset$ and $n = 1$ we obtain the results presented by K. Zima ([4]).

I am greatly indebted to A. Pelczar for reading the paper.

Notation. Let $m, n \in \mathbb{N}$ be fixed in the sequel. For a multiindex $\alpha, \alpha = (\alpha_1, \dots, \alpha_n)$ we put $|\alpha| = \sum_{j=1}^n \alpha_j$ and for a non-negative integer k ,

$$I_k = \{\alpha: |\alpha| = k, \alpha_j \in \{0, 1\} \quad \text{for } j = 1, \dots, n\}.$$

Let $J \subset I = \bigcup_{k=1}^{n-1} I_k$ be fixed. Let $c \in \mathbb{R}^n$ and $c \leq \infty$, which means that $c = (c_1, \dots, c_n)$ where c_i ($i = 1, \dots, n$) are real numbers or positive infinity; if $a \in (-\infty, c)$ is fixed, then $A^* = A_{a,c}^* = (-\infty, c) \setminus [a, c)$. We put for $\mu = 1, 2, \dots; x \in [a, c), x^\mu = (1/\mu(x_1 - a_1), x_2, \dots, x_n)$ and for $h \in [x, c), x(\mu; h) = h^\mu - (0, h_2, \dots, h_n) = (1/\mu(h_1 - a_1), 0, \dots, 0)$. For $\alpha \in I$ and $x \in \mathbb{R}^n$ we put ${}^\alpha x \stackrel{\text{df}}{=} (x_{i_1}, \dots, x_{i_r})$, where $r = |\alpha|$, $\alpha_{i_j} = 1$ ($j = 1, \dots, r$) and

$$\Delta_\alpha(x, y) \stackrel{\text{df}}{=} \prod_{\substack{j \in \{1, \dots, n\} \\ \alpha_j = 0}} [x_j, y_j] \quad \text{for } x, y \in \mathbb{R}^n.$$

We shall write shortly $x^\alpha t^{1-\alpha}$ instead of $(x_1^{\alpha_1} t_1^{1-\alpha_1}, \dots, x_n^{\alpha_n} t_n^{1-\alpha_n})$ and $d^{1-\alpha}t = dt_{i_1}, \dots, dt_{i_s}$, where $\alpha_{i_j} = 0$ ($j = 1, \dots, s$), $s = n - |\alpha|$. For all sets A, B , $A \Delta B \stackrel{\text{df}}{=} (A \setminus B) \cup (B \setminus A)$.

If $A \subset (-\infty, c)$ then we write $y \in C_0(A, R^m)$, if $y: A \rightarrow R^m$ is a continuous function; for $\alpha \in I$ $u \in C_\alpha(A, R^m)$, if u is in A a continuous function with respect to ${}^\alpha x$ and finally $y \in C_J^*(A, R^m)$, if $y \in C_0(A, R^m)$ and for every $\alpha \in J$, $D_\alpha y \in C_\alpha(A, R^m)$.

Let $p, p_\alpha \in \mathbb{N}$, for $\alpha \in J$; we put $q \stackrel{\text{df}}{=} \sum_{\alpha \in J} p_\alpha$. We shall consider mappings

$$E_0: (-\infty, c) \times C_0 \rightarrow R^p, \quad E_\alpha: (-\infty, c) \times C_\alpha \rightarrow R^{p_\alpha}.$$

Assumptions

1. Functions $\psi: A^* \rightarrow R^m$ and $\lambda: [a, c] \rightarrow R^m$ are of class $C_J^*(A^*, R^m)$ and $C_J^*([a, c], R^m)$ respectively and $D_\alpha \lambda|_{A^* \cap [a, c]} = D_\alpha \psi|_{A^* \cap [a, c]}$ for $\alpha \in J \cup I_0$.

2. A function $f: [a, c] \times R^{p+q} \rightarrow R^m$ fulfils:

- (C₁) for every fixed $x \in [a, c]$, the function $f(x, \cdot): R^{p+q} \rightarrow R^m$ is continuous,
- (C₂) for every fixed $y \in R^{p+q}$, the function $f(\cdot, y): [a, c] \rightarrow R^m$ is measurable and continuous with respect to ${}^\alpha x$ in $[a, c]$,
- (C₃) for every $h \in [a, c]$, there exists a function $N_h: [a, h] \rightarrow R^m$, summable in $[a, h]$ and such that $\int_a^x N_h(t) dt < \infty$ and $|f(x, y)| \leq N_h(x)$ for $x \in [a, h]$, $y \in R^{p+q}$.

2'. (W₁) for every $h \in [a, c]$ and $\alpha \in J$, there exists a function $N_h^\alpha: \Delta_\alpha(a, h) \rightarrow R^m$, summable in $\Delta_\alpha(a, h)$ and such that $\int_{\Delta_\alpha(a, x)} N_h^\alpha(1-\alpha t) d^{1-\alpha}t < \infty$ and $N_h(x) \leq N_h^\alpha(1-\alpha x)$ for $x \in [a, h]$.

(W₂) for every $h \in [a, c]$ and $\alpha_0 \in J$, if $\{z_\mu\} \subset R^{p+q}$ and $\{x_\mu\} \subset [a, h]$ such that $z_\mu \rightarrow z$ as $\mu \rightarrow \infty$; $1-\alpha_0 x_\mu = 1-\alpha_0 x$, $\alpha_0 x_\mu \leq \alpha_0 x$ ($\mu = 1, 2, \dots$) and $\alpha_0 x_\mu \rightarrow \alpha_0 x$ as $\mu \rightarrow \infty$, then $\lim_{\mu \rightarrow \infty} f(x_\mu, z_\mu) = f(x, z)$.

3. For every $b \in [a, c]$, there exists $h \in [b, c]$ such that for every $\alpha \in J \cup I_0$:

$$\{x \in [a, h], x_i = b_i, i = 2, \dots, n, \quad u, v \in C_\alpha, u|_{(-\infty, x]} = v|_{(-\infty, x]}\} \Rightarrow \\ \Rightarrow E_\alpha(x, u) = E_\alpha(x, v).$$

4. For every $\alpha \in J \cup I_0$ and $u \in C_\alpha$ the mapping $E_\alpha(\cdot, u): [a, c] \rightarrow R^{p_\alpha}$ is measurable, continuous with respect to ${}^\alpha x$ and finite almost everywhere in every closed subinterval of $[a, c]$.

5. For every $\alpha \in J \cup I_0$, if $\{u^\mu\} \subset C_\alpha$ converges, in a subinterval $(-\infty, b)$ of $(-\infty, c)$, to a function u^0 , uniformly with respect to $1-\alpha x$, then almost everywhere in $(-\infty, b]$, $\lim_{\mu \rightarrow \infty} E_\alpha(x, u^\mu) = E_\alpha(x, u^0)$.

5'. For every $\alpha, \alpha_0 \in J$ and $h \in [a, c]$, if $\{u^\mu\} \subset C_\alpha$ converges in a subinterval $(-\infty, h)$ of $(-\infty, c)$ to a function u^0 , uniformly with respect to $1-\alpha x$ and $\{x_\mu\} \subset [a, h]$ such that $1-\alpha_0 x_\mu = 1-\alpha_0 x$, $\alpha_0 x_\mu \leq \alpha_0 x$ ($\mu = 1, 2, \dots$), $\alpha_0 x_\mu \rightarrow \alpha_0 x$ as $\mu \rightarrow \infty$, then almost everywhere in $[a, h]$, $\lim_{\mu \rightarrow \infty} E_\alpha(x_\mu, u^\mu) = E_\alpha(x, u^0)$.

Problem. We shall consider the following system

$$(1) \quad y(x) = \psi(x) \quad \text{for } x \in A^*$$

$$(2) \quad y(x) = \lambda(x) + \int_a^x f(t, \{E_\alpha(t, D_\alpha y)\}_{\alpha \in J \cup I_0}) dt \quad \text{for } x \in [a, c].$$

Let $b \in (a, c)$. A solution of (1) (2) in $(-\infty, b]$ is any function $y \in C_f^*((-\infty, b], R^m)$, such that for every $\alpha \in J \cup I_0$, $D_\alpha y|_{[a, b]} - D_\alpha \lambda|_{[a, b]}$ is absolutely continuous with respect to ${}^{1-\alpha}x$, fulfilling (1) (2) in the interval $[a, b]$.

We shall prove now the global existence of solutions of problems of the type (1) (2).

Theorem. Under the assumptions 1—5', for every $b \in (a, c)$ there exists at least one solution of (1) (2) in $[a, b]$.

Proof. Let $b \in (a, c)$ be fixed and let $h \in [b, c)$ be chosen for b according to 3. We shall use the notation $x(\mu) = x(\mu, h)$

$$\check{G}(p; \mu) = \left[a, \left(a_1 + (p+1) \frac{1}{\mu} (h_1 - a_1), h_2, \dots, h_n \right) \right] \setminus \\ \setminus \left[a, \left(a_1 + \frac{p}{\mu} (h_1 - a_1), h_2, \dots, h_n \right) \right]$$

$$G(p; \mu) = \overline{\check{G}(p; \mu)} \quad p = 0, \dots, (\mu-1); \mu = 1, 2, \dots$$

Now we define the following sequence of functions. Let $\mu = 1, 2, \dots$

$$(3) \quad y^\mu(x) = \psi(x) \quad \text{for } x \in A^*$$

$$(4) \quad y^\mu(x) = \lambda(x) \quad \text{for } x \in G(0, \mu)$$

$$(5) \quad y^\mu(x) = \lambda(x) + \int_a^{x-x(\mu)} f(t, \{E_\alpha(t, D_\alpha y)\}_{\alpha \in J \cup I_0}) dt \quad \text{for } x \in G(p; \mu), p = 1, \dots, (\mu-1)$$

We shall prove that this definition is correct. Let μ be arbitrarily fixed.

I. In A^* and $G(0; \mu)$, for every $\alpha \in J \cup I_0$ $D_\alpha y^\mu$ is defined by (3) and (4) respectively, in virtue of 1.

II. For $x \in G(1; \mu)$ the variable t , in the integral in (5), varies in $G(0; \mu)$, where for every $\alpha \in J \cup I_0$ $D_\alpha y^\mu$ is defined. In virtue of assumption 3 we take for every $\alpha \in J \cup I_0$ $E_\alpha(t, D_\alpha y^\mu) = E_\alpha(t, y_\alpha^\mu)$ for $t \in G(0, \mu)$, where $y_\alpha^\mu \in C_\alpha$ is an arbitrary extension of $D_\alpha y^\mu$ from the closed set $A^* \cup G(0; \mu)$ onto $(-\infty, c)$. As y^μ is defined for $x \in G(1; \mu)$, hence for every $\alpha_0 \in J \cup I_0$, $D_{\alpha_0} y^\mu$ is defined in $A^* \cup G(0; \mu) \cup G(1; \mu)$, since

$$(6) \quad D_{\alpha_0} y^\mu x = D_{\alpha_0} \lambda(x) + \\ + \int_{D_{\alpha_0}(a, x-x(\mu))} f((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, D_\alpha y)\}_{\alpha \in J \cup I_0}) d^{1-\alpha_0} t \\ \text{for } x \in G(p, \mu), p = 1, \dots, (\mu-1).$$

III. In $A^* \cup [a, h]$, for every $\alpha \in J \cup I_0$ we define $D_\alpha y^\mu$ in a similar manner. Thus (3)—(5) define the sequence $\{y^\mu\} \subset C_f^*(A^* \cup [a, h], R^m)$.

We shall prove that for every $\alpha \in J \cup I_0$ the family $\{D_\alpha y^\mu|_{[a,b]} - D_\alpha \lambda|_{[a,b]}\}$ is a family of uniformly bounded functions and equicontinuous as functions of variables $1-\alpha x$. Indeed, we have

$$(7) \quad |D_{\alpha_0} y^\mu(x) - D_{\alpha_0} \lambda(x)| \\ = |D_{\alpha_0} \int_a^{x-x(\mu)} f(t, \{E_\alpha(t, D_\alpha y)\}_{\alpha \in J \cup I_0}) dt| \leq \int_{\Delta_{\alpha_0}(a,h)} N_h((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}) d^{1-\alpha_0} t \\ \leq \int_{\Delta_{\alpha_0}(a,h)} N_h^{\alpha_0}(1-\alpha_0 t) d^{1-\alpha_0} t.$$

Let $x, \bar{x} \in [a, h]$ be such that $x^\alpha = \bar{x}^\alpha$; we then denote

$$B = \Delta_{\alpha_0}(a, x-x(\mu)) \triangle \Delta_{\alpha_0}(a, \bar{x}-x(\mu)).$$

Now we have

$$(8) \quad |D_{\alpha_0} y^\mu(x) - D_{\alpha_0} \lambda(x) - D_{\alpha_0} y^\mu(\bar{x}) + D_{\alpha_0} \lambda(\bar{x})| \\ = | \int_B f((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, D_\alpha y^\mu)\}_{\alpha \in J \cup I_0}) d^{1-\alpha_0} t| \\ \leq | \int_B N_h((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}) d^{1-\alpha_0} t| \leq \int_B N_h^{\alpha_0}(1-\alpha_0 t) d^{1-\alpha_0} t.$$

Hence for every $\alpha \in J \cup I_0$, the sequence $\{D_\alpha y^\mu|_{[a,b]} - D_\alpha \lambda|_{[a,b]}\}$ satisfies all the conditions of the Arzela theorem as a function of the variables $1-\alpha x$. Thus by the Arzela theorem, choosing step by step subsequences of this sequence, we can obtain a subsequence of indices $\{k^\mu\}$, such that for every $\alpha \in J \cup I_0$ the sequence $\{D_\alpha y^{k^\mu}|_{[a,h]} - D_\alpha \lambda|_{[a,h]}\}$ is convergent in $[a, h]$, uniformly with respect to $1-\alpha x$, to a function ${}^\alpha \tilde{y} \in C_\alpha([a, h], R^m)$. Hence $\{D_\alpha y^{k^\mu}\}$ is convergent in $(-\infty, h]$, uniformly with respect to $1-\alpha x$ to a function

$${}^\alpha y = ({}^\alpha \tilde{y} + D_\alpha \lambda) \cup D_\alpha \psi \in C_\alpha(A^* \cup [a, h], R^m).$$

We put $D_\alpha y^\mu = {}^\alpha y^\mu$ for $\alpha \in J \cup I_0$. By 5 we have $\lim_{\mu \rightarrow \infty} E_\alpha(x, {}^\alpha y^\mu) = E_\alpha(x, {}^\alpha y)$ almost everywhere in $[a, h]$, hence by (C_1) $\lim_{\mu \rightarrow \infty} f(x, \{E_\alpha(x, {}^\alpha y^\mu)\}_{\alpha \in J \cup I_0}) = f(x, \{E_\alpha(x, {}^\alpha y)\}_{\alpha \in J \cup I_0})$ almost everywhere in $[a, h]$. In virtue of the Lebesgue theorem on the integration of the summable and bounded functions, we obtain

$$(9) \quad \lim_{\mu \rightarrow \infty} \int_a^x f(t, \{E_\alpha(t, {}^\alpha y^\mu)\}_{\alpha \in J \cup I_0}) dt = \int_a^x f(t, \{E_\alpha(t, {}^\alpha y)\}_{\alpha \in J \cup I_0}) dt.$$

Besides, for $\alpha_0 \in J \cup I_0$ we have the equality

$$(10) \quad \int_{\Delta_{\alpha_0}(a,x)} f((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, {}^\alpha y^\mu)\}_{\alpha \in J \cup I_0}) d^{1-\alpha_0} t \\ = \int \Delta_{\alpha_0}(a, x-x(\mu)) \dots + \int \Delta_{\alpha_0}(e(\mu), x) \dots,$$

where $e(\mu) = (x_1 - (x(\mu))_1, a_2, \dots, a_n)$. Since $x(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$, by virtue of the obvious estimation and of the theorem on absolutely continuity of integrals, we have

$$(11) \quad | \int_{\Delta_{\alpha_0}(e(\mu), x)} f((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, E_\alpha((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, {}^\alpha y^\mu))_{\alpha \in J \cup I_0}) d^{1-\alpha_0} t| \\ \leq \int_{\Delta_{\alpha_0}(e(\mu), x)} N_h((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}) d^{1-\alpha_0} t \\ \leq \int_{\Delta_{\alpha_0}(e(\mu), x)} N_h^{\alpha_0}(1-\alpha_0 t) d^{1-\alpha_0} t \rightarrow 0 \quad \text{as } \mu \rightarrow \infty.$$

We shall show that $y = y^{(0, \dots, 0)}$ is solution (2). By (5) (9) (10) (11) it follows that

$$(12) \quad y(x) = \lambda(x) + \int_a^x f(t, \{E_\alpha(t, {}^\alpha y)\}_{\alpha \in J \cup I_0}) dt \quad \text{for } x \in [a, h],$$

hence differentiating we obtain for every $\alpha_0 \in J$

$$(13) \quad D_{\alpha_0} y(x) = D_{\alpha_0} \lambda(x) + \int_{\Delta_{\alpha_0}(a, x)} f(x^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha(x^{\alpha_0} t^{1-\alpha_0}, {}^\alpha y)\}_{\alpha \in J \cup I_0}) d^{1-\alpha_0} t.$$

On the other hand, by 5' and (W_2) , we have for every $\alpha_0 \in J$

$$(14) \quad \lim_{\mu \rightarrow \infty} f((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, {}^\alpha y^\mu)\}_{\alpha \in J \cup I_0}) \\ = f(x^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha(x^{\alpha_0} t^{1-\alpha_0}, {}^\alpha y)\}_{\alpha \in J \cup I_0})$$

almost everywhere in $[a, h]$, hence

$$(15) \quad \lim_{\mu \rightarrow \infty} \int_{\Delta_{\alpha_0}(a, x)} f((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha((x-x(\mu))^{\alpha_0} t^{1-\alpha_0}, {}^\alpha y^\mu)\}_{\alpha \in J \cup I_0}) d^{1-\alpha_0} t \\ = \int_{\Delta_{\alpha_0}(a, x)} f(x^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha(x^{\alpha_0} t^{1-\alpha_0}, {}^\alpha y)\}_{\alpha \in J \cup I_0}) d^{1-\alpha_0} t.$$

In virtue of (10) (11) (15), passing in (6) with μ to infinity, we obtain for every $\alpha_0 \in J$

$$(16) \quad {}^{\alpha_0} y(x) = D_{\alpha_0} \lambda(x) + \int_{\Delta_{\alpha_0}(a, x)} f(x^{\alpha_0} t^{1-\alpha_0}, \{E_\alpha(x^{\alpha_0} t^{1-\alpha_0}, {}^\alpha y)\}_{\alpha \in J \cup I_0}) d^{1-\alpha_0} t.$$

Summarizing (13) and (16) we have $D_{\alpha_0} y(x) = {}^{\alpha_0} y(x)$ for $x \in [a, h]$. Hence, by (12), we may easily conclude that y is a solution of (1) (2) in $[a, h]$, since the verification (1) is trivial.

Thus we have proved that function y is a solution of (1) (2) in $[a, b]$, because $b \in [a, h]$, which completes the proof.

REFERENCES

- [1] A. Pelczar, *Some functional differential equations*, Dissert. Math., Warsaw 1973.
- [2] L. Tonelli, *Sulle equazioni funzionali del tipo di Volterra*, Bull. Calcutta Math. Soc. 20 (1928), 31—48.
- [3] L. Tonelli, *Sulle equazioni integrali di Volterra*, Mem. Acc. Bologna (8), 5 (1928).
- [4] K. Zima, *O pewnym układzie równań różniczkowych z opóźnionym argumentem*, Zeszyty Naukowe WSP w Katowicach N° 4 (OG.ZB. N° 21) (1964), 55—61.