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On Mixed Boundary-Value Problems for a Spherical Octant

1. In the paper [1] we solved a mixed boundary value problem of the Neumann and Dirichlet type for the quarter circle. In the present paper we shall construct the solutions of four boundary value problems for the equation

$$(1) \quad \Delta u(X) = 0, \quad X = (x, y, z)$$

in the domain $K = \{(x, y, z): x^2 + y^2 + z^2 < R^2, x > 0, y > 0, z > 0\}$ with boundary Neumann and Dirichlet conditions on the different parts of the boundary.

$$2. \text{ Let us denote: } S = \{(x, y, z): x^2 + y^2 + z^2 = R^2\},$$

$$S_1 = \{(x, y, z): x^2 + y^2 + z^2 = R^2, x > 0, y > 0, z > 0\},$$

$$S_2 = \{(x, y, z): 0 < x < R, y = 0, 0 < z < (R^2 - x^2)^{\frac{1}{2}}\},$$

$$S_3 = \{(x, y, z): x = 0, 0 < y < (R^2 - z^2)^{\frac{1}{2}}, 0 < z < R\},$$

$$S_4 = \{(x, y, z): 0 < x < (R^2 - y^2)^{\frac{1}{2}}, 0 < y < R, z = 0\}.$$

We shall consider four cases. We wish to find functions $u_i(X)$, $i = 1, 2, 3, 4$ harmonic in K satisfying the boundary conditions

$$(2) \quad u_1(X) = f_1(X) \quad \text{for } X \in S_1$$

$$(3) \quad D_y u_1(X) = f_2(x, z) \quad \text{for } X \in S_2$$

$$(4) \quad D_x u_1(X) = f_3(y, z) \quad \text{for } X \in S_3$$

$$(5) \quad D_z u_1(X) = f_4(x, y) \quad \text{for } X \in S_4$$

and

$$(6) \quad u_2(X) = q_1(X) \quad \text{for } X \in S_1$$

$$(7) \quad u_2(X) = q_2(x, z) \quad \text{for } X \in S_2$$

$$(8) \quad u_2(X) = q_3(y, z) \quad \text{for } X \in S_3$$

$$(9) \quad u_2(X) = q_4(x, y) \quad \text{for } X \in S_4$$

and

$$(10) \quad u_3(X) = h_1(X) \quad \text{for } X \in S_1$$

$$(11) \quad u_3(X) = h_2(x, z) \quad \text{for } X \in S_2$$

$$(12) \quad D_x u_3(X) = h_3(y, z) \quad \text{for } X \in S_3$$

$$(13) \quad D_z u_3(X) = h_4(x, y) \quad \text{for } X \in S_4$$

and

$$(14) \quad u_4(X) = p_1(X) \quad \text{for } X \in S_1$$

$$(15) \quad u_4(X) = p_2(x, z) \quad \text{for } X \in S_2$$

$$(16) \quad u_4(X) = p_3(y, z) \quad \text{for } X \in S_3$$

$$(17) \quad D_z u_4(X) = p_4(x, y) \quad \text{for } X \in S_4.$$

The problem of finding a solution of (1) satisfying (2), (3), (4), (5) or (6), (7), (8), (9) or (10), (11), (12), (13) or (14), (15), (16), (17) will be called briefly (D, N, N, N), (D, D, D, D), (D, D, N, N), and (D, D, D, N) respectively.

These problems will be solved by means of Green functions.

3. Given any points $X(x, y, z) = X_1 \in K$, $Y(s, t, w) \in \bar{K}$, where \bar{K} denote the closure of the set K , let $\bar{X}(x, \bar{y}, \bar{z})$ denotes the conjugate point to X with respect to the sphere S . Let $X_2(-x, y, z)$, $X_3(-x, -y, z)$, $X_4(x, -y, z)$, $X_5(x, y, -z)$, $X_6(-x, y, -z)$, $X_7(-x, -y, -z)$ and $X_8(x, -y, -z)$ denote the symmetric images of the points $X_1, X_2, X_3, X_4, X_5, X_6, X_7$ with respect to the corresponding coordinate planes. Let $\bar{X}_i = (e_i^1 \bar{x}, e_i^2 \bar{y}, e_i^3 \bar{z})$ denote conjugate points to $X_i = (e_i^1 x, e_i^2 y, e_i^3 z)$ with respect to S . Let

$$r_i^2 = (s + e_i^1 x)^2 + (t + e_i^2 y)^2 + (w + e_i^3 z)^2,$$

$$\bar{r}_i^2 = (s + e_i^1 \bar{x})^2 + (t + e_i^2 \bar{y})^2 + (w + e_i^3 \bar{z})^2$$

where

$$e_i^1 = \begin{cases} 1 & \text{for } i = 2, 3, 6, 7 \\ -1 & \text{for } i = 1, 4, 5, 8 \end{cases} \quad e_i^2 = \begin{cases} 1 & \text{for } i = 3, 4, 7, 8 \\ -1 & \text{for } i = 1, 2, 5, 6 \end{cases}$$

$$e_i^3 = \begin{cases} 1 & \text{for } i = 5, 6, 7, 8 \\ -1 & \text{for } i = 1, 2, 3, 4 \end{cases}$$

Let $q^2 = x^2 + y^2 + z^2$.

4. First we shall solve the (D, N, N, N) problem.

Let us consider the plane (p) : $As + Bt + Cw = 0$ and the points P_1, P_2 symmetric with respect to (p) and $P_1 \neq P_2$. Let n denote the normal to (p) at the point $Q \in (p)$, $Q = Q(s, t, w)$. Let $d_1^2 = (s - x_1)^2 + (t - y_1)^2 + (w - z_1)^2$, $d_2^2 = (s - x_2)^2 + (t - y_2)^2 + (w - z_2)^2$, (x_i, y_i, z_i) , $i = 1, 2$, being coordinates of the points P_i .

