ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
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Wiesław Niedoba

A Difference Method for a Non-Linear Functional-Differential Equation of the First Order

1. In this paper we shall consider the following functional-differential equation:

$$\frac{\partial u}{\partial t} = f(t, x, u, u_x, u(t, \cdot)) \tag{1.1}$$

where

$$x = (x_1, \ldots, x_p), \qquad u_x = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right)$$

We construct the corresponding difference equation and we prove the convergence of the difference method in the p-dimensional case.

2. We define the following sets:

$$E = \{(t, x)\colon 0 \leqslant t \leqslant d,\ 0 \leqslant x_j \leqslant \sigma,\ (j = 1, \ldots, p)\} \tag{2.1}$$

$$P = \mathop{X}_{i=1}^{p} [0, \sigma] \tag{2.2}$$

$$D = \begin{cases} 0 \leqslant t \leqslant \mathrm{d},\ 0 \leqslant x_j \leqslant \sigma,\ -\infty < u < +\infty \\ -\infty < q_j < +\infty,\ s \in B(P),\ (j = 1, \ldots, p) \end{cases} \tag{2.3}$$

where $B(P)$ is the set of the bounded functions for $x \in P$ with the norm

$$||s|| = \max_{x \in P} |s(x)| \tag{2.4}$$

Let us denote by m the sequence of p natural numbers

$$m = (m_1, \ldots, m_p) \tag{2.5}$$

and let

$$M = (\mu, m), \tag{2.6}$$

where μ is a natural number.

We shall consider the points x^m of the real p dimensional space R^p with coordinates

(2.7) $$x^m = (x_1^{m_1}, \ldots, x_p^{m_p})$$

and also the nodal points

(2.8) $$(t^\mu, x^m) \in R^{p+1}$$

where t^μ and x^m being defined by

(2.9) $$t^\mu = \mu \cdot k, \; x_j^v = v \cdot h$$

$$\mu = 0, 1, \ldots, v = 0, 1, \ldots, j = 1, \ldots, p$$

$0<h = \text{const}, \; 0<k = \text{const}.$

$$\text{for } (t^\mu, x_1^{m_1}, \ldots, x_p^{m_p}) \in E.$$

We define the following sequences of indices

(2.10) $$\omega(M) = (\mu+1, m), \; -j(M) = (\mu, -j(m))$$

where

$$-j(m) = (m_1, \ldots, m_{j-1}, m_j-1, m_{j+1}, \ldots, m_p) \; (j = 1, \ldots, p)$$

Index N_h being chosen so, that

$$N_h \cdot h < \sigma \quad \text{and} \quad (N_h+1) \cdot h \geqslant \sigma$$

Suppose that to each M there corresponds a number v^M. We introduce the following differences and the sum.

$$v^{M\sim} = \frac{1}{k}\left(v^{\omega(M)} - v^M\right)$$

(2.11) $$\overset{*}{v}{}^{M_j} = \frac{1}{h}\left(v^M - v^{-j(M)}\right), \; v^{M\Delta} = \left((v^{M_1}, \ldots, v^{M_p}\right)$$

$$\tilde{v}^\mu(x) = \sum_{M \in Z^\mu} v^M x \sim_{I_M}(x)$$

where

(2.12) $$Z^\mu = \{M\colon 0 \leqslant m_j \leqslant N_h, (j = 1, \ldots, p)\}$$

and

(2.13) $$I_M = \{x\colon m_i h \leqslant x_i < (m_i+1) \cdot h \; (i = 1, \ldots, p)\}$$

and

$$XI_M(x) = \begin{cases} 1, & x \in I_M \\ 0, & x \notin I_M \end{cases}$$

3. In the sequel we shall use the following assumptions H.

Assumptions H

1°. Assume that the function $f(t, x, u, q, s)$, where $x = (x_1, \ldots, x_p)$, $q = (q_1, \ldots, q_p)$ is defined in the region D (2.3) and is of the class C^1 as the function of (t, x, u, q) and of the class C^0 as the functions of s.

2°. The function $u(t, x)$ of the class C^1 satisfies the functional-differential equation

(3.1) $$\frac{\partial u}{\partial x} = f\left(t, x, u, \frac{\partial u}{\partial x}, u(t, \cdot)\right) \quad \text{for } (t, x) \in E$$

where

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right)$$

and the boundary conditions

(3.2) $$u(0, x) = \varphi_0(x) \quad \text{for} \quad x \in P$$

$$u(t, x) = \varphi_j(t, x) \quad \text{for} \quad (t, x) \in E,\ x_j = 0,\ (j = 1, \ldots, p)$$

3°. The derivatives f_{p_j} and the function f fulfil conditions

(3.3) $$f_{q_j} \leqslant 0$$

(3.4) $$f(t, x, u, q, s) - f(t, x, \tilde{u}, q, s) \leqslant L|u - \tilde{u}|$$
$$f(t, x, u, q, s) - f(t, x, u, q, \tilde{s}) \leqslant K\|s - \tilde{s}\|$$

the mesh size h and k being defined so as to obtain

(3.5) $$\sum_{j=1}^{p} f_{q_j} + \frac{h}{k} \geqslant 0 \quad \text{for} \quad (t, x, u, q, s) \in D$$

4. We accept the following boundary conditions for the numbers v^M

(4.1) $$v^M = \begin{cases} \varphi_0^{(x^m)} & \text{for} \quad M = (0, m) \\ \varphi_j(t^\mu, x_1^{m_1}, \ldots, x_j^0, \ldots, x_p^{m_p}) \end{cases}$$

for $\mu = 0, 1, \ldots$ $M = (\mu, m_1, \ldots, 0, \ldots, m_p)$ $(j = 1, \ldots, p)$ the values v^M for the remaining M being defined successively with the aid of the difference equation

(4.2) $$v^{M\sim} = f(t^\mu, x^m, v^M, v^{M\Delta}, \tilde{v}^\mu(\cdot))$$

We denote by u^M the value of solution of equation (3.1) at the nodal point (2.8) and we define the corresponding differences as in case of the numbers v^M. The boundary conditions (3.2) imply the boundary conditions for u^M.

(4.3) $$u^M = \begin{cases} \varphi_0(x^m) & \text{for} \quad M = (0, m) \\ \varphi_j(t^\mu, x_1^{m_1}, \ldots x_j^0, \ldots x_j^{m_p}) & \text{for} \quad \mu = 0, 1, \ldots, \end{cases}$$

$(j = 1, \ldots, p)$ $M = (\mu, m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_p)$
The numbers u^M satisfy the difference equation

(4.4) $$u^{M\sim} = f(t^\mu, x^m, u^M, u^{M\square}, \tilde{u}^\mu(\cdot)) - \eta^M$$

for nodal points $(t^\mu, x^m) \in E$ where $0<\mu$; $1 \leqslant m_j$, $(j = 1, \ldots, p)$ and $\max_m |\eta^M| \to 0$ as $h \to 0$, since $u^M, u^{M\Delta}, \tilde{u}_\mu(\cdot)$ tend to $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$, $u(t, \cdot)$ respectively.

5. Lemma 1. Suppose that the numbers R_μ satisfy the difference inequalities

(5.1) $$R^{\mu\sim} \leqslant L_1 R^\mu + \varepsilon, \quad \mu = 0, 1, \ldots$$

and the initial condition $R^0 = 0$, where

$$R^{\mu\sim} = \frac{1}{H}\left(R^{\mu+1} - R^\mu\right); \ 0<H = \text{const}, 0<L_1 = \text{const}, 0<\varepsilon = \text{const}.$$

Under these assumptions

(5.2) $$R^\mu \leqslant \frac{\varepsilon}{L_1}\left(e^{L_1 H_\mu} - 1\right).$$

This lemma is after Z. Kowalski [1].

6. Lemma 2. Suppose that assumptions H are fulfilled and the values u^M and v^M satisfy (4.3), (4.4) and (4.1), (4.2) respectively at the nodal points in the region E. Let us write

(6.1) $$r^M = u^M - v^M$$

(6.2) $$s^\mu = \max_m r^{\mu,m}, \ z^\mu = \min_m r^{\mu,m}, \ \mu = 0, 1, \ldots$$

Under these assumptions the numbers s^μ and z^μ satisfy respectively the inequalities

(6.3) $$\begin{aligned} s^{\mu\sim} &\leqslant L|r^{\mu,a}| + K\max_{M \in Z^\mu}|r^M| + \varepsilon(h) \\ z^{\mu\sim} &\geqslant -L|r^{\mu,c}| - K\max_{M \in Z^\mu}|r^M| - \varepsilon(h) \end{aligned}$$

where $0 \leqslant \varepsilon(h) \to 0$ as $h \to 0$ and (μ, c), (μ, a) denote certain nodal points.

Proof: There exist $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_p)$ such that

(6.4) $$s^{\mu+1} = \max_m r^{\mu+1,a}$$

(6.5) $$s^\mu = \max_m r^{\mu,m} = r^{\mu,b}$$

whence

(6.6) $$s^{\mu\sim} = \frac{1}{k}\left(r^{\mu+1,a} - r^{\mu,b}\right) = \frac{1}{k}\left(r^{\mu+1,a} - r^{\mu,a}\right) + \frac{1}{k}\left(r^{\mu,a} - r^{\mu,b}\right)$$

Because of (6.1), we have

(6.7) $$\begin{aligned} \frac{1}{k}\left(r^{\mu+1,a} - r^{\mu,a}\right) &= \frac{1}{k}\left(u^{\mu+1,a} - v^{\mu+1,a}\right) - \frac{1}{k}\left(u^{\mu,a} - v^{\mu,a}\right) \\ &= \frac{1}{k}\left(u^{\mu+1,a} - u^{\mu,a}\right) - \frac{1}{k}\left(v^{\mu+1,a} - v^{\mu,a}\right) \end{aligned}$$

If for some j, $1 \leqslant j \leqslant p$, we have $a_j = 0$, inequalities (6.3) hold true, since by the boundary conditions $r^{\mu+1,a} = r^{\mu,a} = 0$ and $r^{\mu,b} \geqslant 0$ by (6.5). For $a_j \geqslant 1$, $(j = 1, \ldots, p)$ however we have

$$(6.8)\quad \frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) = f(t^\mu, x^a, u^{\mu,a}, u^{(\mu,a)\Delta}, \tilde{u}^\mu(\cdot)) + \eta^{\mu,a} - f(t^\mu, x^a, v^{\mu,a}, v^{(\mu,a)\Delta}, \tilde{v}^\mu(\cdot))$$

$$= \eta^{\mu,a} - f(t^\mu, x^a, u^{\mu,a}, u^{(\mu,a)\Delta}, \tilde{u}^\mu(\cdot)) - f(t^\mu, x^a, v^{\mu,a}, u^{(\mu,a)\Delta}, \tilde{u}^\mu(\cdot)) + f(t^\mu, x^a, v^{\mu,a}, u^{(\mu,a)\Delta}, \tilde{u}^\mu(\cdot))$$
$$- f(t^\mu, x^a, v^{\mu,a}, u^{(\mu,a)\Delta}, \tilde{v}^\mu(\cdot)) + f(t^\mu, x^a, v^{\mu,a}, u^{(\mu,a)\Delta}, \tilde{v}^\mu(\cdot)) - f(t^\mu, x^a, v^{\mu,a}, v^{(\mu,a)\Delta}, \tilde{v}^\mu(\cdot)).$$

Applying the mean value theorem to the right — hand member of (6.8) and by the conditions (3.4) and putting $A = (\mu, a)$ we get

$$(6.9)\quad s^{\mu\sim} = \frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}) = \eta^{\mu,a} + L(u^{\mu,a} - v^{\mu,a}) + k\,\|\tilde{u}^\mu(\cdot) - \tilde{v}^\mu(\cdot)\|$$

$$+ \sum_{j=1}^{p} f_{q_j}(\sim)(u^{A_j} - v^{A_j}) + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}) = \eta^{\mu,a} + Lr^{\mu,a} + \frac{1}{k}\sum_{j-1}^{p} f_{q_j}(\sim)$$

$$(u^{\mu,a} - v^{\mu,a} + v^{\mu,-j(a)} - u^{\mu,-j(a)}) + k\max_{M \in Z^\mu} |u^M \chi_{I_M}(x) - v^M \chi_{I_M}(x)| + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}).$$

Now we majorize the right — hand member of (6.9)

From (6.5) follows

$$(6.10)\qquad r^{\mu,-j(a)} \leqslant r^{\mu,b}$$

$$(6.11)\qquad r^{\mu,a} - r^{\mu,-j(a)} \geqslant r^{\mu,a} - r^{\mu,b}$$

and

$$(6.12)\qquad \sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a} - r^{\mu,-j(a)}) \leqslant \sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a} - r^{\mu,b}).$$

since

$$f_{q_j}(\sim) \leqslant 0 \text{ by } (3.3)$$

It follows by (6.9) and (6.12) that

$$(6.13)\quad s^{\mu\sim} \leqslant \eta^{\mu,a} + f_u(\sim) r^{\mu,a} + \frac{1}{h}\sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a} - r^{\mu,b}) + K(\max_{x \in P} \sum_{M \in Z^\mu} |r^M| \chi_{I_M}(x))$$

$$+ \frac{1}{k}(r^{\mu,a} - r^{\mu,b}) = \eta^{\mu,a} + L|r^{\mu,a}| + K(\max_{x \in P} \sum_{M \in Z^\mu} r^M \chi_{I_M}(x)) + \frac{1}{h}(r^{\mu,a} - r^{\mu,b})\left[\sum^{p} f_{q_j}(\sim) + \frac{h}{k}\right]$$

The last term in the right — hand member of (6.13) is non positive by (3.5) and by the nequality $r^{\mu,a} - r^{\mu,b} \leqslant 0$.

We have

$$\max_{x\in P}\Big(\sum_{M\in Z^\mu}|r^M|\chi_{I_M}(x)\Big) = \max_{M\in Z^\mu}|r^M| \tag{6.17}$$

Hence by (6.13) and (6.14) we get the first inequality (6.3)

$$s^{\mu\sim} \leqslant \varepsilon(h)+L|r^{\mu,a}|+K\max_{M\in Z^\mu}|r^M|$$

where $\varepsilon(h) = \max_M|\eta^M|$

The second inequality (6.3) for z^μ can be proved in a similar way. This completes the proof of Lemma 2.

7. The following Lemma 3 is obvious.

Lemma 3. The quantities s^μ, z^μ being defined by (6.2) we have

$$[\max_m r^{\mu,m}]^\sim \leqslant \max(s^{\mu\sim}, -z^{\mu\sim}) \tag{7.1}$$

8. Suppose that the assumptions H are fulfilled and let assume

$$R^\mu = \max_m|r^{\mu,m}| \tag{8.1}$$

Under these assumptions R^μ satisfies the difference inequality

$$R^{\mu\sim} \leqslant (L+K)R^\mu+\varepsilon(h) \tag{8.2}$$

and the initial condition

$$R^0 = 0 \tag{8.3}$$

Proof: $R^0 = 0$ since $r^M = u^M - v^M = 0$ at the boundary nodal points.

From (8.1) and Lemma 3 it follows that

$$R^{\mu\sim} = (\max_m|r^{\mu,m}|)^\sim \leqslant \max_m(s^{\mu\sim}, -z^{\mu\sim}) \tag{8.4}$$

But $s^{\mu\sim}$ and $z^{\mu\sim}$ satisfy the inequalities (6.3), hence

$$R^{\mu\sim} \leqslant \max\Big(L|r^{\mu,a}|+K\sum_{M\in Z^\mu}|r^M|+\varepsilon(h),\ L|r^{\mu,c}|+K\max_{M\in Z^\mu}|r^M|+\varepsilon(h)\Big) \tag{8.5}$$

From the definition (8.1) we obtain

$$|r^{\mu,a}| \leqslant R^\mu, \quad |r^{\mu,c}| \leqslant R^\mu \ \max_{M\in Z^\mu}|r^M| \leqslant R^\mu; \tag{8.6}$$

therefore (8.4), (8.5) and (8.6) imply

$$R^{\mu\sim} \leqslant (L+K)R^\mu+\varepsilon(h)$$

9. Theorem 1. Suppose that assumptions H hold true, u^M is the value of $u(t, x)$ at the nodal point (2.8) and v^M is defined by (4.1), (4.2). Let $\varepsilon(h) = \max_m|\eta^M|$ for all nodal points in the region E.

Under these assumptions we have:
(1) the error estimate

$$r^{\mu,m} \leqslant \frac{\varepsilon(h)}{L+K}(e^{(L+K)k\mu}-1)$$

holds at the nodal points in the set E
(2) the difference method (4.2) is convergent i.e. $\lim\limits_{h\to 0} r^M = 0$

Proof:
(2) follows immediately from (1), from the inequality $k\cdot\mu\leqslant d$ and $\lim\limits_{h\to 0}\varepsilon(h) = 0$.

From Lemma 4 it follows that R^μ satisfies the difference inequalities (8.2) and condition (8.1). Replacing H by K and L_1 by $L+K$ in Lemma 1, we get by (5.2) and (8.2)

$$R^\mu \leqslant \frac{\varepsilon(h)}{L+K}(e^{(L+K)k\mu}-1), \quad \mu = 0, 1, \ldots$$

But

$|r^{\mu,m}| \leqslant R^\mu$, because of (8.1), hence

$$|r^{\mu,m}| \leqslant \frac{\varepsilon(h)}{L+K}(e^{(L+K)k\mu}-1)\,.$$

This completes the proof of Theorem 1.

REFERENCE

[1] Z. Kowalski, *A difference method for a non linear partial differential equation of the first order*, Ann. Polon. Math. 12 (1966).