ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCCXLI PRACE MATEMATYCZNE, ZESZYT 18 1977

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A Difference Method for a Non-Linear Two-Point Boundary Value Problem

Consider a second order nonlinear differential equation of the form

$$F(t, x, x', x'') = 0, \quad t \in [0, 1] \tag{1}$$

together with the classical two-point boundary condition

$$x(0) = a, x(1) = b. \tag{2}$$

We shall replace the derivatives in (1) by suitable difference quotients and thus get the difference equation (cf. (5) below). Using the theorem on a difference inequality proved in [1] we shall show that, under suitable assumptions, solutions of the difference problem (5), (7) approach a solution of the difference problem (1), (2).

1. Let $u = (u_0, \ldots, u_{n+1})$ be a discrete function and let $h>0$. Define for $i = 1, \ldots, n$ the differences as follows

$$u_i^{(1)} = \frac{1}{2h}(u_{i+1} - u_{i-1}) \tag{3}$$

$$u_i^{(2)} = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}). \tag{4}$$

The difference equation we are dealing with is of the form

$$f(t_i, v_i, v_i^{(1)}, v_i^{(2)}) = 0, \quad i = 1, \ldots, n \tag{5}$$

where

$$t_i = ih \quad \text{for} \quad i = 0, \ldots, n+1 \quad \text{and} \quad h = (n+1)^{-1}. \tag{6}$$

Moreover we consider the two-point boundary value condition

$$V_0 = a, \quad V_{n+1} = b. \tag{7}$$

We shall use the following

Assumptions A. The function $f(t, x, y, z) t \in [0, 1], (x, y, z) \in R^3$ continuously differentiable in x, y and z and there are positive numbers η, β and ω such that

(8) $$\frac{\partial f}{\partial x} \leqslant \eta < 0 ,$$

(9) $$\frac{\partial f}{\partial y} \leqslant 2\beta ,$$

(10) $$0 < \omega \leqslant \frac{\partial f}{\partial z}$$

2. Now we prove our main result.

Theorem. Suppose f satisfies Assumptions A. Let x be a solution of (1), (2) and $V = (v_0, \ldots, v_{n+1})$ satisfies (1), (7) with (6).

Suppose moreover that there is Λ (independent of h) such that

(11) $$|x''(t_i) - v_i^{(2)}| \leqslant \Lambda \qquad i = 1, \ldots, n .$$

Then

(12) $$|x(t_i) - v_i| \leqslant \eta^{-1}(\beta h \Lambda + \varepsilon) ,$$

where $\varepsilon = \varepsilon(h)$ (defined below) is such that $\varepsilon(h) \to 0$ as $h \to 0$.

Proof. Set $x_i = x(t_i)$. The discrete function $x = (x_0, \ldots, x_{n+1})$ satisfies the following equation

$$f(t_i, x_i, x_i^{(1)}, x_i^{(2)}) = \varepsilon_i(h), \quad i = 1, \ldots, n .$$

The continuity of f, x' and x'' yields that

$$\varepsilon(h) \to 0 \quad \text{as} \quad h \to 0 ,$$

where

$$\varepsilon(h) = \max\{|\varepsilon_i(h)|;\ i = 1, \ldots, n\} .$$

Set

$$u_i = x_i - v_i , \quad i = 0, \ldots, n+1 .$$

The mean value theorem implies that the discrete function $u = (u_0, \ldots, u_{n+1})$ satisfies the following equation

(13) $$\frac{\partial f}{\partial x}(A_i) u_i + \frac{\partial f}{\partial y}(A_i) u_i^{(1)} + \frac{\partial f}{\partial z}(A_i) u_i^{(2)} = \varepsilon_i$$

where A_i denotes a suitable "intermediate" point set

$$c_i = \frac{\partial f}{\partial x}(A_i)\left(\frac{\partial f}{\partial z}(A_i)\right)^{-1}$$

$$b_i = \frac{\partial f}{\partial y}(A_i)\left(\frac{\partial f}{\partial z}(A_i)\right)^{-1} .$$

Then u_i satisfies the following equation

$$u_i^{(1)}+b_iu_i^{(1)}+c_iu_i = \varepsilon\left(\frac{\partial f}{\partial z}(A_i)\right)^{-1}. \tag{14}$$

Moreover (2) and (7) imply that

$$u_0 = u_{n+1} = 0 . \tag{15}$$

Then (8), (9), (10) and (12) imply that

$$|b_i| \leqslant 2\beta\omega^{-1}$$

$$c_i \leqslant \eta\omega^{-1} < 0$$

$$|u_i^{(2)}| \leqslant \Lambda \quad \text{for} \quad i = 1, \dots, n .$$

Suppose now that for some $i_0 \neq 0$ and $n+1$ $u_{i_0}>0$. Then by (14) and (10) u satisfies for $i = i_0$ the following difference inequality

$$u_i^{(2)}+b_iu_i^{(1)}+c_iu_i \geqslant -\varepsilon\omega^{-1}$$

Applying now Theorem 1 of [1] we infer that

$$u_i \leqslant -\eta^{-1}(\beta h\Lambda+\varepsilon)$$

for all i.

By similar argumentation (using Theorem 2 of [1]) we check that for all i

$$u_i \geqslant \eta^{-1}(\beta h\Lambda+\varepsilon) .$$

This ends the proof.

REFERENCE

[1] K. Szafraniec, *A note on a difference inequality*, Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Prace Matematyczne (to appear).