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On the Existence, Uniqueness and Continuous Dependence of Solution of Boundary Value Problems for Differential Equations with a Delayed Parameter

1. INTRODUCTION

In the present paper, the most general boundary value problem is regarded for differential equations with a delayed parameter. More exactly, we shall consider a functional differential equation of retarded type

$$\dot{x}(t) = f(t, x_t) \tag{1.1}$$

with the boundary condition

$$Ax = \alpha , \tag{1.2}$$

where $f\colon [a, b] \times C^0 \ni (t, \varphi) \to f(t, \varphi) \in R^d$ is a vector function fulfilling the Carathéodory conditions and A is a linear and continuous operator in the space $W^{1,1}$ with values in R^d (for the definitions of C^0 and $W^{1,1}$ see Section 2).

We shall be interested in the question of the existence, uniqueness and continuous dependence of solutions of problems (1.1), (1.2). These questions were discussed by A. Lasota and S. N. Chow in [2] for equations without a delayed parameter.

In Section 4 the theorem on existence and uniqueness will be applied to differential equations with the Lipschitzean right-hand side. The problem of the existence and uniqueness of solutions will be additionally studied for difference equations.

We shall observe that is possible to consider difference equations as a special case of differential equations with a delayed parameter and the right-hand side fulfilling the Carathéodory conditions.

The crucial role in our considerations will be played by Schauder's theorem on open mappings ([3]). The idea of using this theorem in the theory of boundary problems comes from A. Lasota and was first applied by him to the boundary value problem without a delayed parameter in [5] and by S. N. Chow and A. Lasota in [2].

2. DEFINITIONS AND NOTATIONS

Let us consider the space $W^{1,1} = W^{1,1}([a, b], R^d)$ of absolutely continuous functions $x\colon [a, b]\to R^d$ endowed with the norm

$$\|x\|^{1,1}_{[a,b]} = |x(a)| + \int_a^b |\dot{x}(t)|\,dt .$$

For an $h_0>0$ let us set $\Delta = [a-h_0, a]\subset R$. Let $C^0 = C^0(\Delta, R^d)$ denotes the space of continuous functions on Δ with values in R^d. The norm in C^0 is

$$\|x\|^0_\Delta = \sup_{t\in\Delta} |x(t)| .$$

If $x\colon [a-h_0, b]\to R^d$ is a continuous function, then we define

$$x_t(\theta) = \begin{cases} x(a) & \text{for} \quad t+\theta-a\leqslant a \\ & \quad t\in[a, b] \\ x(t+\theta-a) & \text{for} \quad t+\theta-a>a . \end{cases}$$

We recall that the function $f\colon [a, b]\times C^0\to R^d$ fulfils the Carathéodory condition if

(i) for every fixed φ it is measurable with respect to t,

(ii) for every fixed t is continuous with respect to φ and

(iii) for every bounded set $B\subset C^0$ there exists a Lebesgue integrable function m_B such that

$$|f(t, \varphi)|\leqslant m_B(t) \tag{2.1}$$

for $t\in[a, b]$, $\varphi\in B$.

Let Ψ be a given family of functions $f\colon [a, b]\times C^0\to R^d$. We say that Ψ satisfies the Carathéodory condition uniformly if every function $f\in\Psi$ fulfils conditions (i), (ii) and

(iii') for every bounded set $B\subset C^0$ there exists a Lebesgue integrable function M_B such that

$$|f(t, \varphi)|\leqslant M_B(t)$$

for every $t\in[a, b]$, $\varphi\in B$ and $f\in\Psi$.

We say that a sequence $\{f_n\}$ ($f_n\colon [a, b]\times C^0\to R^d$) converges continuously to a function $f\colon [a, b]\times C^0\to R^d$ if for every sequence $\{\varphi_n\}\subset C^0$ convergent to a function $\varphi\in C^0$ in the space C^0 we have

$$f_n(t, \varphi_n)\to f(t, \varphi) ,$$

as $n\to\infty$ for every $t\in[a, b]$.

By a solution in Carathéodory's sense of equation (1.1) we mean the continuous function $x\colon [a-h_0, b]\to R^d$ which is absolutely continuous on the interval $[a, b]$, satisfies (1.1) almost everywhere on $[a, b]$, and is constant on Δ.

Condition (H^0): The function $f\colon [a, b]\times C^0\to R^d$ fulfils condition (H^0) if for every function $\varphi\in C^0$ there is exactly one solution x of equation (1.1) which satisfies the initial condition $x_a = \varphi$.

We denote by $\mathscr{L} = \mathscr{L}(W^{1,1}([a, b], R^d))$ the space of all continuous linear operators from $W^{1,1}$ to R^d with the norm topology. Throughout the paper, Ω will denote an open subset of $\mathscr{L}$.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH A DELAYED PARAMETER

We shall prove the following.

Theorem 3.1. Let a fixed function $f\colon [a,b]\times C^0\to R^d$ satisfies simultaneously the Carathéodory condition and condition (H^0). Assume that for every operator $A\in\Omega$ and every vector $\alpha\in R^d$ boundary value problem (1.1), (1.2) has at most one solution. Then, for every $A\in\Omega$ and every $\alpha\in R^d$ problem (1.1), (1.2) has exactly one solution.

Proof. Consider the mapping $g\colon R^d\ni k\to x^k$, where x^k is a solution of equation (1.1) fulfilling the initial condition $x_a^k = k$. By condition (H^0) all solutions of (1.1) have the property of continuous dependence upon the initial values ([4]). Thus g is continuous.

For a given operator $A\in\Omega$ we define the mapping $F\colon R^d\to R^d$ in the following manner

$$F(k) = Ax^k .$$

The assumptions of the theorem assure the continuity and injection of F. It follows from Schauder's theorem on open mappings [3] that the range of F, $F(R^d)$, is open in the space R^d (¹).

To prove the theorem, it suffices to show that for every operator $A\in\Omega$ and for every vector $\alpha\in R^d$ the function F is surjective.

Suppose not. Then $A_1\in\Omega$ and $\alpha_1\in R^d$, and a mapping $F_1\colon R^d\to R^d$ can be found such that $F_1(R^d)\neq R^d$.

Let us take the boundary point $w\notin F_1(R^d)$. Then there exists a sequence $\{s_n\}\subset R^d$ such that $F_1(s_n)\in F_1(R^d)$ and $F_1(s_n)\to w$, as $n\to\infty$. If the sequence $\{s_n\}$ were convergent to a certain $\bar{s}\in R^d$, then by the continuity of F_1, $F_1(s_n)\to F_1(\bar{s})$, as $n\to\infty$, whence $F_1(\bar{s}) = w$. This contradicts the assumption that $w\notin F_1(R^d)$. Suppose then, that the sequence $\{s_n\}$ is not convergent. Then there exists a number $\varepsilon>0$ such that

$$|s_{n+m}-s_n|\geqslant\varepsilon \quad \text{for} \quad n = 1, 2, \ldots, m = m(n)\geqslant 1 .$$

By the definition of F_1 we have

$$F_1(s_{n+m})-F_1(s_n) = A_1(x^{s_{n+m}}-x^{s_n}) = A_1 y_n ,$$

where $y_n = x^{s_{n+m}}-x^{s_n}$. The functions $x^{s_{n+m}}$ and x^{s_n} are two distinct solutions of the same equation and fulfil the inequality

$$\|y\|_{[a,b]}^{1,1}\geqslant|y_n(a)| = |s_{n+m}-s_n|\geqslant\varepsilon .$$

On the other hand

$$|F_1(s_{n+m})-F_1(s_n)|\to 0, \quad \text{as} \quad n\to\infty ,$$

since the sequence $\{F_1(s_n)\}$ is convergent. From this we obtain

$$\frac{|F_1(s_{n+m})-F_1(s_n)|}{\|y\|{\to}_{[a,b]}^{1,1}}\to 0, \quad \text{as} \quad n\to\infty .$$

(¹) Schauder's theorem regarding the Banach space reads as follows:

Let U be an open subset of a Banach space E and $F\colon U\to E$ be an injection. If F is a complete continuous vector field on U, then $F(U)$ is an open set in E.

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where $y_n = x^{s_{n+m}}-x^{s_n}$. The functions $x^{s_{n+m}}$ and x^{s_n} are two distinct solutions of the same equation and fulfil the inequality

$$||y||_{[a,b]}^{1,1}\geqslant|y_n(a)| = |s_{n+m}-s_n|\geqslant\varepsilon\,.$$

On the other hand

$$|F_1(s_{n+m})-F_1(s_n)|\to 0, \quad \text{as} \quad n\to\infty\,,$$

since the sequence $\{F_1(s_n)\}$ is convergent. From this we obtain

$$\frac{|F_1(s_{n+m})-F_1(s_n)|}{||y||\to_{[a,b]}^{1,1}}\to 0, \quad \text{as} \quad n\to\infty\,.$$

(¹) Schauder's theorem regarding the Banach space reads as follows:

Let U be an open subset of a Banach space E and $F\colon U\to E$ be an injection. If F is a complete continuous vector field on U, then $F(U)$ is an open set in E.

Thus we can find a linear and continuous operator $A_{(n)}$ (dependent on the choice of n), which reaches a value equal to $F_1(s_{n+m})-F_1(s_n)$ on the function y_n. More exactly, we have

$$A_{(n)}y_n = A_1 y_n$$

and

$$\|A_{(n)}\|^{1,1} = \frac{|F_1(s_{n+m})-F_1(s_n)|}{\|y\|^{1,1}_{[a,b]}} \to 0, \quad \text{as} \quad n\to\infty$$

where

$$\|A_{(n)}\|^{1,1} = \sup\{|A_{(n)}x|: \|x\|^{1,1}_{[a,b]} = 1\}.$$

Since $A_1 \in \Omega$, we can choose $\bar{n}$ so large that $A_{\bar{n}}-A_1 \in \Omega$. But

$$0 = (A_{\bar{n}}-A_1)y_{\bar{n}} = (A_{\bar{n}}-A_1)(x^{s_{\bar{n}}+\bar{m}}-x^{s_{\bar{n}}}),\ \bar{m} = m(\bar{n}).$$

Thus it is evident that the boundary value problem

$$\dot{x}(t) = f(t, x_t),$$

$$(A_{\bar{n}}-A_1)x^{s_{\bar{n}}+\bar{m}} = (A_{\bar{n}}-A_1)x^{s_{\bar{n}}} = \alpha$$

has two distinct solutions for $A_{\bar{n}}-A_1 \in \Omega$. This contradicts the uniqueness assumptions, which completes the proof of the theorem.

4. APPLICATION OF THE THEOREM ON EXISTENCE AND UNIQUENESS

A particular case of the boundary condition (1.2) is the Cauchy condition

(4.1) $$x_a = \alpha, \quad \alpha \in R^d.$$

Indeed, (1.2) reduces to (4.1) for

(4.2) $$A_0 x = x(a).$$

If the function f in Theorem 3.1 fulfils condition (H^0), then problem (1.2), (4.2) has exactly one solution. Theorem 4.1 below will show that "close" (in the sense of the topology in $\mathscr{L}$) problems to the Cauchy problem have the same property.

Theorem 4.1. Let us suppose that a fixed function f: $[a, b]\times C^0\to R^d$ fulfils the Carathéodory condition and the Lipschitz condition

(4.3) $$|f(t, \varphi)-f(t, \psi)| \leq K(t)\|\varphi-\psi\|^0_A$$

for all functions $\varphi, \psi \in C^0$ and $t \in [a, b]$, where K is a certain non-negative integrable function. Then for every operator A belonging to the open ball $B(A_0, \eta)$, and for every vector $\alpha \in R^d$ the boundary value problem (1.1), (1.2) has exactly one solution, where A_0 is given by (4.2) and

(4.4) $$\eta = (1+e^{\int_a^b K(\tau)d\tau}\int_a^b K(t)dt)^{-1}.$$