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## On Some Comparison Theorems for a Functional Inequality Connected with Cosine

§ 1. In the present paper we shall deal with the functional inequality

$$(1) \quad \psi(x+y) \leq \psi(x)\psi(y) - \sqrt{1-\psi^2(x)}\sqrt{1-\psi^2(y)},$$

where  $\psi$  is an unknown real function fulfilling the condition

$$(2) \quad \psi(0) = 1.$$

As has been proved in [3] continuous solutions  $\psi$  of (1) satisfying (2) are either constant functions or there is a positive  $a$  such that  $\psi$  is positive in  $(0, a)$ .

Thus in the sequel we shall study the following class of functions:

**Definition.** Let  $a > 0$  be a fixed number. A function  $\psi: [0, a] \rightarrow [0, 1]$  is said to belong to the class  $\mathcal{A}$  if it is a solution of (1), continuous in  $[0, a]$  and such that

$$(3) \quad \psi(a) = 0 \quad \text{and} \quad \psi(x) > 0 \quad \text{for} \quad x \in (0, a).$$

§ 2. We shall need some results from [3], which we quote here as lemmas.

**Lemma 1.** If  $\psi \in \mathcal{A}$ , then

$$(4) \quad \psi(x) = \cos f(x), \quad x \in [0, a]$$

where  $f$  is a superadditive function, i.e.

$$(5) \quad f(x+y) \geq f(x) + f(y)$$

fulfilling the conditions

$$(6) \quad f: [0, a] \rightarrow [0, \pi/2].$$

$$(7) \quad f(0) = 0 \quad \text{and} \quad f(a) = \pi/2$$

**Lemma 2.** Let  $\psi \in \mathcal{A}$  and let  $\varphi$  be the continuous solution of the equation

$$(8) \quad \varphi(x+y) = \varphi(x)\varphi(y) - \sqrt{1-\varphi^2(x)}\sqrt{1-\varphi^2(y)}$$

such that

$$(9) \quad \varphi(0) = 1 \quad \text{and} \quad \varphi(a) = 0.$$

If there exists  $x_0 \in (0, a]$  such that

$$\psi(x_0) = \varphi(x_0)$$

then

$$(10) \quad \psi\left(\frac{x_0}{2^n}\right) \geq \varphi\left(\frac{x_0}{2^n}\right).$$

Lemma 3. If  $\psi \in A$ ,  $\varphi$  is a continuous solution of equation (8) fulfilling conditions (9) and

$$(11) \quad \psi\left(\frac{a}{2^n}\right) = \varphi\left(\frac{a}{2^n}\right) \quad \text{for} \quad n=1, 2, \dots$$

then

$$\psi(x) = \varphi(x) \quad \text{for} \quad x \in [0, a].$$

Moreover the general continuous solution of equation (8) satisfying conditions (9) is given by the formula

$$(12) \quad \psi(x) = \cos cx, \quad \text{where} \quad c = \pi/2a$$

(see [2]).

§ 3. In the present section we prove two comparison theorems for solutions of inequality (1).

Theorem 1. If  $\psi \in A$  and if there exists the limit

$$(13) \quad c = \lim_{r \rightarrow 0} \frac{\arccos \psi(r)}{r}$$

and

$$(14) \quad 0 < c \leq \pi/a$$

then

$$(15) \quad \psi(x) \leq \cos cx \quad \text{for} \quad x \in [0, a]$$

and

$$(16) \quad c \in \left(0, \frac{\pi}{2a}\right]$$

Proof. It follows from Lemma 1 that the function  $f(x) = \arccos \psi(x)$  is superadditive. We see from (13) that

$$\lim_{r \rightarrow 0} \frac{f(r)}{r} = c$$

There also exists the limit

$$\lim_{r \rightarrow 0} \frac{f(rx)}{r} = cx$$

Since  $\frac{1}{n} \rightarrow 0$  and  $f$  fulfils (5) and (6), then

$$f(x) \geq \frac{f(x/n)}{1/n} \rightarrow cx$$

Since  $c \in \left(0, \frac{\pi}{a}\right]$  and  $x \in [0, a]$ , then  $cx \in [0, \pi]$  and

$$\cos f(x) \leq \cos cx \quad \text{for } x \in [0, a].$$

Therefore we obtain (15), by virtue of (4).

If  $c \in \left(\frac{\pi}{2a}, \frac{\pi}{a}\right]$  then from the inequality (15) it follows for  $x = a$  that

$$\psi(a) \leq \cos ca < \cos \frac{\pi}{2} = 0$$

which contradicts (2). This ends the proof.

**Theorem 2.** Let the assumptions of theorem 1 be fulfilled. If moreover

$$(17) \quad c = \pi/2a$$

then

$$(18) \quad \psi(x) = \cos cx = \varphi(x) \quad \text{for } x \in [0, a].$$

**Proof.** Inequality (15) is fulfilled by virtue of theorem 1. Condition (17) implies (9), whence we obtain

$$(19) \quad \psi\left(\frac{a}{2^n}\right) \geq \varphi\left(\frac{a}{2^n}\right) \quad n = 1, 2, \dots,$$

by virtue of Lemma 2, because  $x_0 = a$ . Inequalities (19) and (15) imply condition (11), and by virtue of Lemma 3 we have (18).

**§ 4.** Here we shall prove another comparison theorem for which we need the following result from [1].

**Lemma 4.** If  $g$  is a superadditive function fulfilling the conditions (5)—(7), then there exists an extension  $g^*$  of the function  $g$  on the interval  $(0, \infty)$  which is a superadditive function and we have

$$(20) \quad g^*(x) = g(x) \quad \text{for } x \in [0, a].$$

If  $\psi \in A$ , then it follows from Lemma 1 that  $f(x) = \arccos \psi(x)$  is a superadditive function and that the conditions (5)—(7) are fulfilled. Hence by virtue of Lemma 4 there exists a superadditive function  $f^*$  which is an extension of the function  $f$  into  $(0, \infty)$ .

**Theorem 3.** Let  $\psi \in A$ . If there exists a positive limit

$$(21) \quad 0 < \lim_{r \rightarrow \infty} \frac{f^*(r)}{r} = d \leq \frac{\pi}{a}$$

