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Note on an Implicit Function Theorem in a Non-differentiable Case

1. Introduction. Recently Shui-Nee Chow and A. Lasota [1] have proved an implicit function theorem using the concept of multivaluated derivative. Not changing in fact the idea of the proof we shall show that the implicit function theorem holds true with more general conditions on a function appearing in the equation considered (see the example at the end of this paper).

2. Notations and definitions. Let E be a Banach space with norm $|\cdot|$. Let $\mathrm{cocl}(E)$ denote the set of all convex closed non-empty subsets of E. For $A \in \mathrm{cocl}(E)$, $|A| = \sup\{|x|: x \in A\}$.

A map $P: E \to \mathrm{cocl}(E)$ is called homogeneous if for every real λ, $P(\lambda x) = \lambda P(x)$, for all $x \in E$.

Let U be an open set of E, $p: U \to E$ and $x_0 \in U$. A map $P: E \to \mathrm{cocl}(E)$ is called an upper derivative of p at x_0 if P is homogeneous and $p(x) - p(x_0) \in P(x - x_0) + \varepsilon(|x - x_0|)$ where $\varepsilon: R \to E$ and $|\varepsilon(|x - x_0|)|/|x - x_0| \to 0$ as $x \to x_0$.

Let $x \in E$, $A \in \mathrm{cocl}(E)$, then $d(x, A)$ will denote $\inf\{|x - y|: y \in A\}$. $B(r)$ will denote a ball in E with center 0 and radius r.

3. Implicit function theorem. Let us quote Theorem 1 [1]: Given a functional equation

$$(*) \qquad x = p(x) - q(x, v)$$

where $p: U \to E$, $q: U \times B(r) \to E$ are completely continuous, $r > 0$ and $q(x, v) \to 0$ uniformly in x as $|v| \to 0$. Suppose there exists $x_0 \in U$ such that $x_0 = p(x_0)$ and there exists an upper derivative P of p at x_0 such that P is upper semicontinuous compact [1]) and $x \in P(x) \Rightarrow x = 0$. Then there exist $\bar{r} \in (0, r]$ and $R > 0$ such that

(1) for every $v \in B(\bar{r})$ there is a solution x_v of equation (*) which satisfies $|x_v - x_0| < R$; and

(2) $|x_v - x_0| \to 0$ as $|v| \to 0$.

In our paper we replace the conditions that P is compact upper semicontinuous and $x \in P(x) \Rightarrow x = 0$ by one more general condition.

(¹) A map $P: E \to \mathrm{cocl}(E)$ is upper semicontinuous if the graph $\{(x, y): y \in P(x)\}$ is closed in $E \times E$. A map $P: E \to \mathrm{cocl}(E)$ is compact if for every bounded $V \subset E$ the set $\{y: y \in P(x), x \in V\}$ has bounded closure.

Theorem. Given a functional equation

$$(*) \qquad x = p(x)+q(x, v)$$

where $p: U \to E$ and $q: U \times B(r) \to E$ are completely continuous, $r>0$ and $q(x, v) \to 0$ uniformly in x as $|v| \to 0$. Suppose there exists $x_0 \in U$ such that $x_0 = p(x_0)$ and that there exists an upper derivative P of p at x_0 such that

$$\alpha = \inf\{d(x, P(x)), x \in E, |x| = 1\} > 0 .$$

Then there exist $\bar{r} \in (0, r]$ and $R>0$ such that

(1) for every $v \in B(\bar{r})$ there is a solution x_v of equation $(*)$ which satisfies $|x_v - x_0| < R$, and

(2) $|x_v - x_0| \to 0$ as $|v| \to 0$.

We can prove a result similar to that of [1, Lemma 1]:

Proposition. Let $P: E \to \mathrm{cocl}(E)$ be a homogeneous map and

$$\alpha = \inf\{d(x, P(x)), x \in E, |x| = 1\} .$$

If $\alpha > 0$ then

$$x \in P(x)+b \Rightarrow |x| \leqslant \frac{1}{\alpha}|b| \quad \text{for all} \quad x, b \in E .$$

Proof. For $x = 0$ the Proposition is evidently true. Let $x \in P(x)+b$, $x \neq 0$. Since P is homogeneous, we have

$$\frac{x}{|x|} \in P\left(\frac{x}{|x|}\right)+\frac{b}{|x|} .$$

It means $x/|x| = \bar{x}+b/|x|$, $\bar{x} \in P(x/|x|)$. Hence

$$\left|\frac{b}{|x|}\right| = \left|\frac{x}{|x|}-\bar{x}\right| \geqslant d\left(\frac{x}{|x|}, P\left(\frac{x}{|x|}\right)\right).$$

Since $|x/|x|| = 1$, we have $|b/|x|| \geqslant \alpha$ which is equivalent to $|x| \leqslant \frac{1}{\alpha}|b|$.

4. A more precise analysis of the proof of Theorem 1 [1] shows that the continuity of the solution may be proved by our Proposition, whereas for the existence of the solution the conditions that P is compact upper semicontinuous and $x \in P(x) \Rightarrow x = 0$ are not necessary, so this part of Theorem 1 [1] may be shown in the same way as in [1]. Nevertheless we include both proofs in detail for the convenience of the reader.

Let $\alpha > 0$. Since P is an upper derivative of p at x_0 then for $\alpha/2$ there exists $R>0$ such that

$$|\varepsilon(|x-x_0|)|/|x-x_0| < \frac{\alpha}{2} \quad \text{for all} \quad |x-x_0| < R .$$

The condition $q(x, v) \to 0$ uniformly in x as $|v| \to 0$ implies the existence of a decreasing real function $\delta = \delta(s)$ such that $\delta(s) \geqslant 0$ for each s, $\delta(s) \to 0$ as $s \to 0$ and

$$|q(x, v)| < \delta(|v|) \quad \text{for all} \quad |x-x_0| < R .$$

Suppose now x_v is a solution of (*) and $|x_v - x_0| < R$, $v \in B(r)$, then

$$x_v - x_0 = p(x_v) + q(x_v, v) - p(x_0) \in P(x_v - x_0) + \varepsilon(|x_v - x_0|) + q(x_v, v).$$

Hence, by the Proposition, we have

$$|x_v - x_0| \leqslant \frac{1}{\alpha} |\varepsilon(|x_v - x_0|) + q(x_v, v)| \leqslant \tfrac{1}{2}|x_v - x_0| + \frac{1}{\alpha}\delta(|v|).$$

Therefore

$$|x_v - x_0| \leqslant \frac{2}{\alpha}\delta(|v|) \to 0 \quad \text{as} \quad |v| \to 0.$$

Let

$$\varphi_v(u) = u - p(x_0 + u) + p(x_0) - q(x_0 + u, v).$$

Since $\delta(|v|) \to 0$ as $|v| \to 0$, there exists $\bar{r}$ such that $\frac{2}{\alpha}\delta(|v|) < R$ as $v \in B(\bar{r})$. We shall show that

$$\varphi_v(u) \neq \lambda \varphi_v(-u)$$

for every $\lambda \in [0, 1]$, $v \in B(\bar{r})$ and $|u| = \frac{2}{\alpha}\delta(|v|)$.

If this is not the case, then there exist $\lambda \in [0, 1]$, $v \in B(\bar{r})$ and u such that

$$\varphi_v(u) = \lambda \varphi_v(-u), \qquad |u| = \frac{2}{\alpha}\delta(|v|).$$

Hence

$$u = \frac{1}{1+\lambda}\big(p(x_0+u) - p(x_0)\big) + \frac{\lambda}{1-\lambda}\big(p(x_0) - p(x_0 - u)\big) + \\ + \frac{1}{1+\lambda} q(x_0 + u, v) + \frac{\lambda}{1+\lambda}\big(-q(x_0 - u, v)\big).$$

Since P is homogeneous and convex, we have

$$u \in P(u) + \varepsilon(|u|) + \frac{1}{1+\lambda} q(x_0 + u, v) + \frac{\lambda}{1+\lambda}\big(-q(x_0 - u, v)\big).$$

Therefore, from the Proposition

$$|u| \leqslant \frac{1}{\alpha}\left(|\varepsilon(|u|)| + \frac{1}{1+\lambda}|q(x_0+u, v)| + \frac{\lambda}{1+\lambda}|q(x_0 - u, v)|\right) \leqslant \\ \leqslant \frac{1}{\alpha}\big(|\varepsilon(|u|)| + \delta(|v|)\big) < \tfrac{1}{2}|u| + \frac{1}{\alpha}\delta(|v|).$$

Thus $|u| < \frac{2}{\alpha}\delta(|v|)$, which contradicts our assumption. Now by applying Borsuk's theorem to $\varphi_v(u)$ we obtain u_v such that $|u_v| < R$ and $\varphi_v(u_v) = 0$. By setting $x_v = u_v + x_0$ we get the required solution of (*).

In this proof we have utilized Borsuk's antipodal fixed point theorem in the following form: