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Existence of Solutions of Some Cauchy-Darboux Problems for Partial Differential-Functional Equations

The subject of this paper is the existence of solutions of some partial differential-functional equations with initial-boundary conditions of the Cauchy-Darboux type and some integro-differential-functional equations.

In order to prove existence theorems we have used the classical Schauder fixed-point theorem.

The bibliography of partial differential equations of hyperbolic type and partial differential-functional equations is very rich. We give here only a short list of references. General Cauchy-Darboux problems were considered for differential equations without deviations the arguments, for instance in [2]; we refer also to the bibliography cited in [2]. The Darboux problem for two-dimentional partial differential-functional equations was discussed for example in [1], [4], [5]. Some n-dimentional problems of the Darboux type for partial differential-functional equations and partial differential equations were considered for instance in [6] and [3]. Very general initial-boundary problems were considered in [7] and [8]; in [8] the fixed-point theorem of Schauder was used for the existence of solutions of such problems.

Chapter I

NOTATION. PRELIMINARY DEFINITIONS AND FUNDAMENTAL ASSUMPTIONS

1. Preliminary notation

By R, R_*, N, N_0 we denote — respectively — the sets of numbers: real, real non-negative, positive integers, non-negative integers. For set A and $n \in N$ we denote by A^n — as usual — the Cartesian product $A \times A \times \ldots \times A$ (n-times); thus

$$R^n = \{(x_1, \ldots, x_n) \colon x_i \in R, i = l, \ldots, n\}$$

is identified with the n-dimensional Euclidean space. We put R^n_* and N^n_0 instead of $(R_*)^n$ and $(N_0)^n$. For set $A \subset R^n$ we denote by ∂A, int A and $\bar{A}$, the boundary, the interior and the closure of A, respectively.

If $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in R^n$, then we put

$$a \leqslant b \overset{\text{df}}{\Leftrightarrow} a_i \leqslant b_i \quad \text{for every} \quad i \in \{1, \dots, n\}, \tag{1.1}$$

$$a <^0 b \overset{\text{df}}{\Leftrightarrow} a_i < b_i \quad \text{for every} \quad i \in \{1, \dots, n\}, \tag{1.2}$$

$$a < b \overset{\text{df}}{\Leftrightarrow} a \leqslant b \quad \text{and} \quad a_i < b_i \quad \text{for some} \quad i \in \{l, \dots, n\}. \tag{1.3}$$

For $a, b \in R^n$ such that $a \leqslant b$, we put

$$[a, b] = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] \tag{1.4}$$

and if $a < b$

$$[a, b) = [a_1, b_1) \times [a_2, b_2) \times \dots \times [a_n, b_n). \tag{1.5}$$

For $\mu \in R^n_*$ we put

$$|\mu| \overset{\text{df}}{\Leftrightarrow} \mu_1 + \dots + \mu_n. \tag{1.6}$$

We shall consider the following sets:

$$I^*_n = \{\mu = (\mu_1, \dots, \mu_n) \in R^n\colon \mu_i \in \{0, 1\} \quad \text{for} \quad i \in \{1, \dots, n\}\} \tag{1.7}$$

and

$$I_n = \{\mu \in I^*_n\colon |\mu| \leqslant n-1\}. \tag{1.8}$$

For any $x \in R^k$ we denote by $||x||$ the Euclidean norm of x; we shall write $||\cdot||$ for every dimensions k.

2. Conventions concerning multi-indices

Let m and n be positive integers; hence forward we shall consider m and n as fixed throughout this paper.

We shall consider μ, called multi-index, belonging to the set

$$I = I_n$$

or — possibly — to the set

$$I^* = I^*_n,$$

where I_n and I^*_n are given by (1.8) and (1.7) for our fixed n. We adopt the following convention: the letter μ without any assumption everywhere denotes a multi-index belonging to I; if not, it will be noted explicitly. Thus the assumption

$$\mu \in I$$

will be dropped very often in the sequel, whenever the condition

$$\mu \in I^*$$

will be explicitly assumed.

For $\mu \in I^*$ we put

(2.1) $$\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n) \quad \text{with} \quad \hat{\mu}_i = 1 - \mu_i \; (i = 1, \ldots, n).$$

Setting

(2.2) $$\beta^i_j = \beta^i_j(\mu) = \delta_{ij}\hat{\mu}_i, \text{ where } \delta_{ij} = 0 \text{ if } i \neq j, \delta_{ij} = 1 \text{ if } i = j$$

and denoting successively those of multi-indices

$$\beta^1 = (\beta^1_1, \ldots, \beta^1_n), \ldots, \beta^i = (\beta^i_1, \ldots, \beta^i_n), \ldots, \beta^n = (\beta^n_1, \ldots, \beta^n_n)$$

which are different from $(0, \ldots, 0)$, by

$$\gamma^1(\mu), \gamma^2(\mu), \ldots$$

we may write

(2.3) $$\hat{\mu} = \sum_{i=1}^{n} \beta^i = \sum_{i=1}^{n-|\mu|} \gamma^i(\mu)$$

3. Set Δ

Suppose that $b \in R^n_*$ is fixed and such that

$$0 = (0, \ldots, 0) <^0 b = (b_1, \ldots, b_n).$$

Let Δ be a subset of R^n_*, fixed throughout the paper, such that:

(3.1) Δ is closed and $\operatorname{int} \Delta$ as well as Δ are connected,

(3.2) $$\begin{cases} (\{b_1\} \times (-\infty, b_2] \times \ldots \times (-\infty, b_n]) \cap R^n_* \subset \Delta, \\ ((-\infty, b_1] \times \{b_2\} \times (-\infty, b_3] \times \ldots \times (-\infty, b_n]) \cap R^n_* \subset \Delta, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ ((-\infty, b_1] \times \ldots \times (-\infty, b_{n-1}] \times \{b_n\}) \cap R^n_* \subset \Delta, \end{cases}$$

and

(3.3) $$\Delta \subset [0, b].$$

In order to formulate further conditions assumed for the set Δ, we need certain notations and definitions.

By S we denote the set

(3.4) $$\Delta \cap \overline{([0, b] \setminus \operatorname{int} \Delta) \setminus \{x \in R^n : \exists_i x_i = b_i\}}$$

It is clear that $\partial\Delta \subset S$ and S is compact, since Δ is compact as a closed subset of the compact set $[0, b]$. We suppose that S, being obviously a hypersurface in R^n, has the following form

(3.5) $$S = \Sigma^0 \cup \bigcup \Sigma_i$$

where

(3.6) $$\Sigma_i = S \cap \{x \in R^n : x_i = 0\} \quad (i = 1, \ldots, n)$$

and Σ^0 is a hypersurface, the properties of which we shall describe precisely below. Let us denote by proj_k the following mapping:

(3.7) $$\text{proj}_k\colon R^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in R^{n-1}.$$

In particular

(3.8) $$\text{proj}_k(\Sigma_k') = \{(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in R^{n-1}\colon (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) \in \Sigma_k\}.$$

It is assumed that there are n functions φ_i ($i = 1, \ldots, n$) having the following properties:

(3.9) $$\varphi_i\colon S_i \to [0, b_i]$$

where

(3.10) $$S_i \overset{\text{df}}{\Leftrightarrow} \overline{\text{proj}_i(S) \setminus \text{proj}_i(\Sigma_i)} \quad (i = 1, \ldots, n),$$

(3.11) $$\varphi_i(x) = 0 \quad \text{for} \quad x \in S_i \cap \text{proj}_i(\Sigma_i) \quad (i = 1, \ldots, n),$$

(3.12) $$\varphi_i \quad \text{is continuous} \quad (i = 1, \ldots, n),$$

(3.13) every φ_i is strictly decreasing with respect to each variable, that is: if $j \in \{1, \ldots, n\} \setminus \{i\}$ and $x_j < y_j$ then
$$\varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)$$
$$> \varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n),$$

(3.14) the following conditions are equivalent:
$$x_j^0 = \varphi_j(x_1^0, \ldots, x_{j-1}^0, x_{j+1}^0, \ldots, x_n^0) \text{ for some } j \in \{1, \ldots, n\}$$
$$x_k^0 = \varphi_k(x_1^0, \ldots, x_{k-1}^0, x_{k+1}^0, \ldots, x_n^0) \text{ for each } k \in \{1, \ldots, n\},$$

(3.15) $$\Sigma^0 = \{(x_1, \ldots, x_n) \in S\colon x_k = \varphi_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \text{ for every } k \in \{1, \ldots, n\}\}.$$

Remark 3.1. In virtue of the condition (3.14) we have.

(3.16) $$\Sigma^0 = \{(x_1, \ldots, x_n) \in S \text{ there is } i \in \{1, \ldots, n\} \text{ such that } x_i = \varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\}.$$

For $x \in \text{proj}_i(R^n) \subset R^{n-1}$ we put:

(3.17) $$\tilde{\varphi}_i(x) = \begin{cases} \varphi_i(x) & \text{for} \quad x \in S_i \\ 0 & \text{for} \quad x \notin S_i. \end{cases}$$

Moreover we put

(3.18) $$h \overset{\text{df}}{=} \max\{(b_i - \tilde{\varphi}_i(\text{proj}_i(b)))\colon i \in \{1, \ldots, n\}\}.$$

4. Sets G, G^0, B and B^0

For a point x belonging to the set Δ we put

(4.1) $$\Delta_x = \Delta \cap [0, x].$$

Observe that

$$\Delta_b = \Delta.$$

Suppose that there are two bounded and closed subsets G and G^0 of the space R^n (fixed in the sequel) such that

(4.2) $$G \subset G^0$$

(4.3) $$G \cap \Delta = G \cap \Sigma^0$$

and

(4.4) $$G^0 \cap \Delta = G^0 \cap S .$$

Now we put

(4.5) $$B = G \cup \Delta$$

(4.6) $$B^0 = G^0 \cup \Delta .$$

Hypothesis H^0. The sets B and B^0 are closures of their interiors:

(4.7) $$B = \overline{\operatorname{int} B} \qquad B^0 = \overline{\operatorname{int} B^0} .$$

5. Modulus of continuity

A function

$$\delta\colon R_* \to R_*$$

will be called a modulus of continuity of a function g defined in a subset A of a metric space X (provided with the metric ϱ), having its values in a Banach space Y (with a norm $\|\cdot\|$), if and only if

(5.1) δ is non-decreasing and $\delta(t) \to 0$ as $t \to 0$,

(5.2) $$\sup\{\|g(x)-g(y)\|\colon \varrho(x,y) \leqslant t\} < \delta(t)$$
for $t \in R_*$, $x, y \in A$.

In the sequel we shall consider the Euclidean norm $\|\cdot\|$; see Sec. 1.

6. Some spaces of functions

If E is a bounded region in R^n, then we denote by $C^k(\bar{E})$ the vector space of all functions

$$z\colon \bar{E} \to R^m$$

possessing in E ($= \operatorname{int} E$) the partial derivatives

(6.1) $$D^\mu z = \frac{\partial^{|\mu|} z}{\partial x^\mu} = \frac{\partial^{\mu_1+\ldots+\mu_n}}{\partial x_1^{\mu_1} \ldots \partial x_n^{\mu_n}} z \qquad (\mu \in I)$$

such that these derivatives have their finite limits in each point x^0 belonging to ∂E as $x \in \operatorname{int} E$ tends to x^0. Furthermore we require that putting for $x^0 \in \partial E$

$$D^\mu z(x^0) \overset{\mathrm{df}}{=} \lim_{\substack{x \to x^0 \\ x \in E}} D^\mu z(x)$$

we obtain for each function z belonging to $C^k(\bar{E})$, the continuous functions

$$D^\mu z\colon \bar{E}\to R^m, \quad \mu\in I.$$

If F is a subset of R^n and $E\subset F$, then by $C^k(\bar{E}; F)$ we denote the space $C^k(\bar{E})\cap C^0(F)$ (where $C^0(F)$ denotes the set of all continuous mappings from F into R^m) provided with the norm

$$(6.2) \qquad \|z\|_k = \sup\{\|z(x)\|\colon x\in F\} + \sum_{\substack{0<|\mu|<k \\ \mu\in I}} \sup\{\|D^\mu z(x)\|\colon x\in \bar{E}\}.$$

The symbol $C^k(\bar{E})\cap C^0(F)$ is clearly understood as the symbol of the set of those functions from F into R^m which are continuous and have restrictions to the set $\bar{E}$ belonging to $C^k(\bar{E})$. The space $C^k(\bar{E})$ is identified with $C^k(\bar{E}; E)$.

7. Operators A_μ

Suppose that there is given a family of operators $\{A_\mu\}_{\mu\in I}$:

$$(7.1) \qquad A_{(0,\dots,0)}\colon C^{n-1}(B; B^0)\to C^{n-1}(\Delta)$$

$$(7.2) \qquad A_\mu\colon C^{n-1-|\mu|}(B)\to C^{n-1-|\mu|}(\Delta) \quad \text{for} \quad \mu>0,$$

for which we admit the following

Hypothesis H_1.

(I) A_μ are bounded i.e. for every positive number d and every μ, there exists a number $c_\mu(d)\in R_*$ such that for $z\in C^{n-1-|\mu|}(B)$ $\big(z\in C^{n-1}(B; B^0)$ if $\mu=0=(0,\dots,0)\big)$ satisfying the inequality

$$\|z\|_{n-1-|\mu|}\leqslant d$$

we have

$$(7.3) \qquad \|A_\mu z\|_{n-1-|\mu|}\leqslant c_\mu(d).$$

(II) A_μ are uniformly equicontinuous i.e. for every positive number ε there exists a positive number $\eta=\eta(\varepsilon)$ such that if $z,\tilde{z}\in C^{n-1-\mu}(B)$ $(z,\tilde{z}\in C^{n-1}(B; B^0)$ if $\mu=0$ and $\|z-\tilde{z}\|_{n-1-|\mu|}\leqslant\eta$

then

$$(7.4) \qquad \|A_\mu z - A_\mu\tilde{z}\|_{n-1-|\mu|}\leqslant\varepsilon \quad \text{for} \quad \mu\in I.$$

(III) If $\{Z^\mu\}_{\mu\in I}$ is a set of some families Z^μ of functions such that

$$Z^\mu\subset C^{n-1-|\mu|}(B),\ Z^0\subset C^{n-1}(B; B^0)$$

and there exists a modulus of continuity δ common to all functions of all families Z^μ, then there exists a modulus of continuity η_δ common to all functions of all families

$$A_\mu(Z^\mu)=\{A_\mu(z)\colon z\in Z^\mu\}\subset C^{n-1-\mu}(\Delta);$$

this means that

$$(7.5) \qquad \|(A_\mu z)(x)-(A_\mu z)(y)\|_{n-1-|\mu|}\leqslant\eta_\delta(t) \quad \text{for} \quad z\in Z^\mu$$

$$\text{whenever } x,y\in\Delta,\ t\in R_*,\ \|x-y\|\leqslant t.$$

Remark 7.1. It is easy to see that the operators

$$A_\mu = \text{Identity ("restricted" to } \Delta)$$

satisfy trivially Hypothesis H_1.

8. Functions λ and ψ

Let λ and ψ be some vector-valued functions:

$$\lambda\colon \Delta \to R^m, \qquad \psi\colon G^0 \to R^m$$

fixed in the sequel.
We assume that λ and ψ satisfy the following

Hypothesis H_2.

(I) For each $\mu \in I$, $\mu > 0$, there exist continuous derivatives $D^\mu \lambda$ and $D^\mu \psi$ in $\operatorname{int}\Delta$ and $\operatorname{int} G$ — respectively — as well as their limits as we approach to any point on $\partial\Delta$ and ∂G respectively; these limits we denote also by $D^\mu \lambda$ and $D^\mu \psi$. If $\operatorname{int} G = \emptyset$ then the assumptions concerning the existence of $D^\mu \psi$ in $\operatorname{int} G$ are omitted.

(II) For every $\mu \in I$, $\mu > 0$ and every $x \in G \cap \Delta$ we have

$$(8.1) \qquad (D^\mu \lambda)(x) = (D^\mu \psi)(x)$$

which means precisely that

$$(8.2) \qquad \lim_{\substack{y \to x \\ y \in \operatorname{int}\Delta}} (D^\mu \lambda)(y) = \lim_{\substack{y \to x \\ y \in \operatorname{int} G}} (D^\mu \psi)(y).$$

If $\operatorname{int} G = \emptyset$ then the condition (8.1) is omitted.

(III) For $x \in G^0 \cap \Delta$ the following equality

$$(8.3) \qquad \lambda(x) = \psi(x)$$

is satisfied.

(IV) The derivative $D^{(1,\dots,1)}$ exists and is continuous in $\operatorname{int}\Delta$ and has a finite limit at each point of the boundary of Δ.

Remark 8.1. From (I) and (IV) it follows that the function λ has all derivatives $D^\mu \lambda$ in the set Δ for $\mu \in I^*$.

Chapter II

FORMULATION OF PROBLEMS FOR INTEGRAL AND DIFFERENTIAL-FUNCTIONAL EQUATIONS

9. Problem I

Suppose that B and B^0 fulfil Hypothesis H^0. Let λ and ψ and operators A_μ be as in Chapter I; in particular we assume Hypotheses H_1 and H_2.
We put

$$(9.1) \qquad k = (2^n - 1)m.$$