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Analytic Continuation of Pluriharmonic Functions

ABSTRACT

In this paper we shall present a construction of the pluriharmonic envelope of analyticity and of holomorphy for a region in R^{2n}. We shall prove that this envelope is "invariant" with respect to biholomorphic transformations of the region. We shall also construct the polyharmonic envelope of analyticity and of holomorphy for a polycylindrical region.

INTRODUCTION

Let U be an open set in R^n; by $A(U)$ we denote the space of all real analytic functions of n real variables on U.

Let Ω be open in C^n; by $\mathcal{O}(\Omega)$ we denote the space of all holomorphic functions on Ω.

By $\mathcal{O}_n$ we denote the sheaf of germs of holomorphic functions on C^n, and by π_n the natural projection $\mathcal{O}_n \to C^n$.

Every region (i.e. a non-empty, open and connected set) in $\mathcal{O}_n$ will be called an analytic function.

An analytic function $F \subset \mathcal{O}_n$ will be called arbitrarily continuable if:

(*) $\forall z \in \pi_n(F)$, $\forall F_z \in \pi_n^{-1}(z) \cap F$ and for every continuous mapping $\gamma\colon I = [0, 1] \to \pi_n(F)$ such that $\gamma(0) = z$, there exists a continuous mapping $\hat{\gamma}\colon I \to F$ such that $\hat{\gamma}(0) = F_z$ and $\pi_n \circ \hat{\gamma} = \gamma$.

It is known that (*) is equivalent to

(**) $\forall z \in \pi_n(F)$, $\forall F_z \in \pi_n^{-1}(z) \cap F$, $\forall \varphi \in F_z$: φ may be holomorphically extended on every polydisc $P(z; r) \subset \pi_n(F)$.

Fix a region D in R^n, $n \geqslant 2$, and a vector subspace S in $A(D)$. We consider the two following problems:

(A) Whether there exists a set Ω in C^n such that:

($A1$) Ω is a connected domain of holomorphy containing D;

($A2$) for every $f \in S$ there exists an arbitrarily continuable analytic function F over Ω (i.e. $\pi_n(F) = \Omega$) such that $\forall x \in D$ the germ f_x belongs to F;

($A3$) there exists a function $f_0 \in S$ such that its continuation F_0 (in the sense of ($A2$)) has the following property:

$\forall z \in \Omega$, $\forall (F_0)_z \in \pi_n^{-1}(z) \cap F_0$, $\forall \varphi \in (F_0)$: φ cannot be holomorphically extended on any polydisc $P(z; R)$ if $P(z; R)\setminus\Omega \neq \emptyset$.

(H) Whether there exists a set Ω in C^n such that:

($H1$) Ω satisfies ($A1$);

($H2$) $\forall f \in S\, \exists \tilde{f} \in \mathcal{O}(\Omega)$: $\tilde{f}|_D = f$;

(H3) there exists $f_0 \in S$ such that its holomorphic continuation $\tilde{f}_0$ on Ω cannot be holomorphically continued beyond Ω.

Remarks. The function F in ($A2$) is uniquely determined by f.

The solution of (A) is uniquely determined by D and S, so if Ω satisfies (A), we write $\Omega = D_S^A$ and we call D_S^A the S — envelope of analyticity for D.

Similarly, the solution of (H) is uniquely determined by D and S, we write $\Omega = D_S^H$ and we call D_S^H the S — envelope of holomorphy for D.

If (H) has the solution, then (A) has the solution and $D_S^A = D_S^H$.

If (A) has a solution which satisfies ($H2$), then (H) has the solution and $D_S^H = D_S^A$.

If (A) has a solution which is homotopically simply connected, then D_S^A satisfies ($H2$).

If Ω satisfies ($A1$), ($A2$) and ($H3$) then $\Omega = D_S^A$.

It is possible to show that for $S = A(D)$ the problem (A) has not any solution. On the other hand, if S is too small (for example, if $S = R[x_1, \ldots, x_n]|_D$), then ($H$) has the only solution $D_S^H = C^n$; this case is not interesting.

We shall present some solutions of (A) and (H) for particular cases of S.

1° $S = H(D)$ — the space of all real harmonic functions on D. P. Lelong proved that here the answer to the problem (A) is always positive and that in the case $n = 2p \geq 4$, $p \in N$, the answer to the problem (H) is also positive. More exactly we have

Lelong's theorem [4]. Given $z = (z_1, \ldots, z_n) \in C^n$, put

$$T(z) = \{t = (t_1, \ldots, t_n) \in R^n : \sum_{j=1}^{n} (t_j - z_j)^2 = 0\}.$$

Set $\tilde{D} = \{z \in C^n : \exists a \in D, \exists \gamma: I \to C^n$ such that γ is continuous, $\gamma(0) = a$, $\gamma(1) = z$ and $\forall \tau \in I: T(\gamma(\tau)) \subset D\}$.

Then $\tilde{D} = D_{H(D)}^A$ and in the case $n = 2p \geq 4$: $\tilde{D} = D_{H(D)}^H$ (see [4], theorems 2, 4 and 6).

Note that in the cases $n = 2$ and $n = 2p+1$, $p \in N$, there exist examples of regions D for which $\tilde{D}$ is not any solution of (H) (see [4], p. 15, also [2] theorem 6).

2° $S = H_L(D)$ — the space of all solutions of a linear elliptic differential operator L with constant coefficients. C. O. Kiselman in [3] proved that for every convex region D there exists a maximal convex region Ω in C^n which satisfies ($H1$) and ($H2$).

In Section 1 of this paper we answer the problems (A) and (H) for a region $D \subset R^{2n}$ and for $S = PH(D)$ — the space of all pluriharmonic functions on D. This will be an extension of Lelong's theorem and of theorems 2, 3, 4, 7 from [2] (see also [4], p. 17). In Section 2 we consider the problems (A) and (H) for the space $H_\alpha(D)$ consisting of all polyharmonic functions of the given type on a polycylindrical region D.

1. ANALYTIC CONTINUATION OF PLURIHARMONIC FUNCTIONS

In this section D denotes a region in R^{2n}. For $z \in C^k$ and the positive numbers $r_1, \ldots, r_k$, by $P(z; r_1, \ldots, r_k)$ we denote the polydisc in C^k with the center z and the radii $r_1, \ldots, r_k$. If $r_1 = \ldots = r_k = r$ we write $P(z; r; k)$ instead of $P(z; r, \ldots, r)$.

For $z = (z_1, z_2, \ldots, z_{2n-1}, z_{2n}) \in C^{2n}$ set

(1)
$$\phi(z) \overset{\mathrm{df}}{=} (z_1 + iz_2, \ldots, z_{2n-1} + iz_{2n}) \in C^n .$$

Let

(2)
$$\hat{D} = \{z \in C^{2n} \colon \phi(z),\ \phi(\bar{z}) \in D\} ;$$

we identify R^{2n} with C^n.

Remarks. $\hat{D}$ is a region in C^{2n} symmetric with respect to the mapping $C^{2n} \ni z \to \bar{z} \in C^{2n}$, $\hat{D} \cap R^{2n} = D$; we identify $R^{2n} \times \{0\} \subset C^{2n}$ with C^n.

D is starlike with respect to $\xi \in D$ if and only if $\hat{D}$ is starlike with respect to ξ.

D is convex if and only if $\hat{D}$ is convex.

D is homotopically simply connected if and only if $\hat{D}$ is homotopically simply connected.

The following theorem (analogical to Lelong's theorem) plays the fundamental role in our considerations.

Theorem 1. $\hat{D}$ satisfies $(A2)$ for $S = PH(D)$, moreover for every $f \in PH(D)$ the analytic arbitrarily continuable continuation F of f over $\hat{D}$ has the single-valued real part on $\hat{D}$ and

(3)
$$Re F(z) = \tfrac{1}{2}\big(f(\phi(z)) + f(\phi(\bar{z}))\big),\ z \in \hat{D} .$$

Proof. Let $f \in PH(D)$ be fixed. Locally in D, f is the real part of a holomorphic function, so there exists an analytic arbitrarily continuable function $G \subset \mathcal{O}_n$ over D such that $f = Re G$.

We shall give a construction of the continuation of f over $\hat{D}$. Let $z \in \hat{D}$, take $G_{\Phi(z)} \in \pi_n^{-1}(\phi(z)) \cap G$ and $G_{\Phi(\bar{z})} \in \pi_n^{-1}(\phi(\bar{z})) \cap G$. Let $\varphi \in G_{\Phi(z)}$, $\varphi \in \mathcal{O}\big(P(\phi(z); \varrho; n)\big)$, $P(\phi(z); \varrho; n) \subset D$, $\psi \in G_{\Phi(\bar{z})}$, $\psi \in \mathcal{O}\big(P(\phi(\bar{z}); \varrho; n)\big)$, $P(\phi(\bar{z}); \varrho; n) \subset D$. Set

(4)
$$\lambda(w) = \tfrac{1}{2}\big(\varphi(\phi(w)) + \overline{\psi(\phi(\overline{w}))}\big),\ w \in P(z; \tfrac{1}{2}\varrho; 2n) .$$

$P(z; \frac{1}{2}\varrho; 2n) \subset \hat{D}$, so λ is well defined and $\lambda \in \mathcal{O}\big(P(z; \frac{1}{2}\varrho; 2n)\big)$. We take the germ λ_w of λ at $w \in P(z; \frac{1}{2}\varrho; 2n)$. Now we change, if possible, $w \in P(z; \frac{1}{2}\varrho; 2n)$, $G_{\Phi(z)} \in \pi_n^{-1}(\phi(z)) \cap G$, $G_{\Phi(\bar{z})} \in \pi_n^{-1}(\phi(\bar{z})) \cap G$ and $z \in \hat{D}$. The set of all germs of the type λ_w, obtained in this way, we denote by F. It is obvious that F is an arbitrarily continuable analytic function over $\hat{D}$ which extends f. Since $f = Re G$, (4) implies (3). This completes the proof.

The mapping $C^n \ni z \to (\phi(z), \phi(\bar{z})) \in C^n \times C^n$ is a homeomorphism and its inverse mapping A is given by the formula

(5)
$$C^n \times C^n \ni \big(\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n)\big) \mapsto$$
$$\to \left(\frac{\xi_1 + \bar{\eta}_1}{2}, \frac{\xi_1 - \bar{\eta}_1}{2i}, \ldots, \frac{\xi_n + \bar{\eta}_n}{2}, \frac{\xi_n - \bar{\eta}_n}{2i}\right) \in C^{2n} .$$

Analogically to Theorem 2 in [2], we can prove the following

Lemma 1. A function $h \in PH(D)$ has a holomorphic continuation $\hat{h}$ on $\hat{D}$ if and only if there exists $f \in \mathcal{O}(D)$ such that $h = Re f$, moreover:

$$f(\xi) = h(\xi) + i(2 Im \hat{h}(A(\xi, \eta)) + \text{const.}), \ \xi \in D; \ \eta \in D \text{ fixed}; \tag{6}$$

$$\hat{h}(z) = \tfrac{1}{2}\left(f(\phi(z)) + \overline{f(\phi(\bar{z}))}\right), z \in \hat{D}. \tag{7}$$

Proposition 1. If D is homotopically simply connected then $\hat{D}$ satisfies ($H2$) for $S = PH(D)$.

Proof. If D is homotopically simply connected then every function from $PH(D)$ is the real part of a holomorphic function from $\mathcal{O}(D)$, so we can use Lemma 1.

The following theorem is analogous to Theorem 7 in [2].

Theorem 2. Let D, G be regions in C^n, $f = (f_1, \ldots, f_n)$: $D \to G$ be biholomorphic; $f_k = u_k + iv_k$, $\hat{u}_k$, $\hat{v}_k$ denote the corresponding holomorphic continuations of u_k and v_k on $\hat{D}$, $k = 1, \ldots, n$; $\hat{f} \overset{\text{df}}{=} (\hat{u}_1, \hat{v}_1, \ldots, \hat{u}_n, \hat{v}_n)$: $\hat{D} \to C^{2n}$. Then $\hat{f}(\hat{D}) = \hat{G}$ and $\hat{f}$: $\hat{D} \to \hat{G}$ is biholomorphic.

Proof. The proof is analogical as in the case $n = 1$.

By Lemma 1:

$$\hat{u}_k(z) = \tfrac{1}{2}\left(f_k(\phi(z)) + \overline{f_k(\phi(\bar{z}))}\right),$$

$$\hat{v}_k(z) = \frac{1}{2i}\left(f_k(\phi(z)) - \overline{f_k(\phi(\bar{z}))}\right), \ z \in \hat{D}, \quad k = 1, \ldots, n.$$

Set, for $z \in C^{2n}$,

$$\hat{T}(z) = \{\phi(z), \phi(\bar{z})\}. \tag{8}$$

It is easy to show that for every $z \in \hat{D}$ $\hat{T}(\hat{f}(z)) = f(\hat{T}(z))$, so $\hat{f}(\hat{D}) \subset \hat{G}$.

Now, let $w \in \hat{G}$ be fixed. There exist $\xi, \eta \in D$ such that $\phi(w) = f(\xi)$, $\phi(\bar{w}) = f(\eta)$, so $w = \hat{f}(A(\xi, \eta))$, (see (5)), hence $\hat{G} \subset \hat{f}(\hat{D})$.

For the mapping $g = f^{-1}$: $G \to D$ we construct $\hat{g}$ (in the same way as $\hat{f}$ for f). Then $\hat{f}$, $\hat{g}$ are holomorphic, $(\hat{f} \circ \hat{g})|_G = id_G$, $(\hat{g} \circ \hat{f})|_D = id_D$, so $\hat{g} = (\hat{f})^{-1}$. This completes the proof.

Corollary 1. Let D, G, f be as in Theorem 2. Then $\hat{D}$ satisfies ($H2$) if and only if $\hat{G}$ satisfies ($H2$).

If $\hat{D}$ satisfies ($H2$), $\hat{D}$ need not satisfy ($H3$). For example, if we take $D = \Omega \setminus K$ such that:

(a) Ω is a region in C^n,

(b) $K \subset \Omega$, K is a non-empty compact set,

(c) D is homotopically simply connected, then $\hat{\Omega}$ satisfies ($H2$) for $PH(D)$ but $\hat{D} \not\supseteq \hat{\Omega}$.

Now we shall discuss situations when $\hat{D}$ is the solution of (A) or (H).

Theorem 3. If D is a domain of holomorphy in C^n then $\hat{D}$ is the solution of (A).

Proof. By Theorem 1 it suffices to show that $\hat{D}$ satisfies ($H3$).

Let $f \in \mathcal{O}(D)$ be a function which cannot be holomorphically continued beyond D. Let $\hat{h}$ be given by the formula (7). It suffices to show that $\hat{h}$ cannot be holomorphically continued beyond $\hat{D}$.

Suppose that there exist $z \in \hat{D}$, $r>0$ and $\varphi \in \mathcal{O}(P(z; r; 2n))$ such that $P(z; r; 2n) \setminus \hat{D} \neq \phi$ and φ is equal to $\hat{h}$ in a neighbourhood of z. It is easy to prove that $\phi(P(z; r; 2n)) = P(\phi(z); 2r; n)$, $\psi(P(z; r; 2n)) = P(\phi(\bar{z}); 2r; n)$, where $\psi(z) = \phi(\bar{z})$, $z \in C^{2n}$. We have $P(\phi(z); 2r; n) \setminus D \neq \phi$ or $P(\phi(\bar{z}); 2r; n) \setminus D \neq \phi$; suppose, for example, that $P(\phi(z); 2r; n) \setminus D \neq \phi$.

Set $g(\xi) = 2\varphi(A(\xi, \phi(\bar{z}))) - \overline{f(\phi(\bar{z}))}$, $\xi \in P(\phi(z); 2r; n)$. g is well defined, $g \in \mathcal{O}(P(\phi(z); 2r; n))$ and g is equal to f in a neighbourhood of $\phi(z)$. Since f cannot be continued beyond D, this gives a contradiction. This completes the proof.

Conversely, we have

Theorem 4. If $\hat{D}$ satisfies $(A1)$ then D is a domain of holomorphy.

Proof. We shall use the following well known theorem (see [1], Theorem 2.5.14): Let Ω and Ω' be holomorphy domains in C^n and in C^m, respectively, and let u be a holomorphic map of Ω into C^m. Then $\Omega_u = \{z \in \Omega: u(z) \in \Omega'\}$ is a domain of holomorphy.

In our situation we set, for fixed $\eta \in D$, $m = 2n$, $\Omega = C^n$, $\Omega' = \hat{D}$, $u(\xi) = A(\xi, \eta)$, $\xi \in C^n$. Then $\Omega_u = D$, so D is a domain of holomorphy. The proof is completed.

Note that if we put $\Omega = C^{2n}$, $\Omega' = D \times D^*$, $u(z) = (\phi(z), \overline{\phi(\bar{z})})$, $z \in C^{2n}$, where $D^* = \{\xi \in C^n: \bar{\xi} \in D\}$, then from the assumption that D is a domain of holomorphy, we may deduce that $\hat{D}$ is a domain of holomorphy. Hence the essential meaning of Theorem 3 is such that $\hat{D}$ is a domain of holomorphy with respect to the space of all holomorphic functions in $\hat{D}$ which are the continuations of functions from $PH(D)$.

Theorems 1, 3 and 4 imply

Corollary 2. $\hat{D}$ is the solution of (A) if and only if D is a domain of holomorphy.

Corollary 3. $\hat{D}$ is the solution of (H) if and only if D is a domain of holomorphy and

$$\text{(R)} \qquad \forall h \in PH(D)\ \exists f \in \mathcal{O}(D): \ h = Re f.$$

Corollary 4. If D is not any domain of holomorphy, D satisfies (R) and the envelope of holomorphy Ω of D is univalent, then $\hat{\Omega}$ is the solution of (H) for $PH(D)$.

Directly from the definitions of T (see Lelong's theorem) and — of $\hat{T}$ (see (8)) we have:

$$\text{(9)} \qquad \hat{T}(z) \subset T(z_1, z_2) \times \ldots \times T(z_{2n-1}, z_{2n}) \subset T(z), \ z = (z_1, z_2, \ldots, z_{2n-1}, z_{2n}) \in C^{2n}.$$

Whence $\tilde{D} \subset \hat{D}$ (more exactly — $\{z \in C^{2n}: T(z) \subset D\} \subset \hat{D}$) and for $D = D_1 \times \ldots \times D_n$, D_i — a region in C, $i = 1, \ldots, n$, $\hat{D} = \tilde{D}_1 \times \ldots \times \tilde{D}_n$ (in the case $n = 1$: $\hat{D} = \tilde{D}$).

Below we shall give an example of a situation when $\tilde{D} \not\supseteq \hat{D}$. Let $D = B = \{\xi \in R^{2n}: |\xi| < r\}$ —

the ball in R^{2n}, $n \geqslant 2$. It is possible to show that $\tilde{B} = \{z \in C^{2n}: t(z) < r\}$, where for $z = x + iy \in C^{2n}$: $t(z) = (|x|^2 + |y|^2 + 2\sqrt{|x|^2|y|^2 - \langle x, y\rangle^2})^{1/2}$, see [3], [5] also [2]. Let $\theta \in (2, 2\sqrt{2})$, $z = \frac{r}{\theta}((1, 1, 0, \ldots, 0) + i(0, 0, 1, 1, 0, \ldots, 0))$. It is easy to check $z \in \hat{B} \setminus \tilde{B}$.

2. ANALYTIC CONTINUATION OF POLYHARMONIC FUNCTIONS

Fix $k\in N$, $\alpha=(\alpha_1,\dots,\alpha_k)\in N^k$ and let Ω be an open set in $R^{|\alpha|}=R^{\alpha_1}\times\dots\times R^{\alpha_k}$.

The function $u\colon \Omega\to R$ is called α-polyharmonic if for every $a=(a_1,\dots,a_k)\in\Omega$ the function $x_i\to u(a_1,\dots,a_{i-1},x_i,a_{i+1},\dots,a_k)$ is harmonic in a neighbourhood of a_i, $i=1,\dots,k$.

By $H_\alpha(\Omega)$ we denote the space of all α-polyharmonic functions on Ω.

We consider the problems (A) and (H) for $S=H_\alpha(D)$, where $D=D_1\times\dots\times D_k$, D_i is a region in R^{α_i}, $i=1,\dots,k$.

First, note that in this case we can reduce the problem to the case $\alpha_i\geqslant 2$, $i=1,\dots,k$. Further we always make this assumption.

The main result of this section is the following

Theorem 5. The set $\tilde{D}_1\times\dots\times\tilde{D}_k$ is the α-polyharmonic envelope of analyticity for D. Moreover, if $\alpha_i=2p_i\geqslant 4$, $p_i\in N$, $i=1,\dots,k$, then $\tilde{D}_1\times\dots\times\tilde{D}_k$ is the α-polyharmonic envelope of holomorphy for D.

Proof. Obviously $\tilde{D}_1\times\dots\times\tilde{D}_k$ is a domain of holomorphy, so $(A1)=(H1)$ is satisfied.

By iteration of the classical integral representation with the Newton kernel for harmonic functions we obtain an integral representation for α-polyharmonic functions; more exactly we get the following

Lemma 2. Let E_i denote the Newton kernel in R^{α_i}. Set $E(x)=E_1(x_1)\dots E_k(x_k)$, $x=(x_1,\dots,x_k)\in R^{|\alpha|}$, $x_i\neq 0$, $i=1,\dots,k$. Let G_i be a region in R^{α_i} such that $\bar{G}_i\subset D_i$, ∂G_i is the union of a finite number of surfaces of class C^1, $i=1,\dots,k$. Let $f\in H_\alpha(D)$. Then, for every $x=(x_1,\dots,x_k)\in G_1\times\dots\times G_k$:

$$f(x)=\int_{\partial G_1}\dots\int_{\partial G_k} W_\alpha(f,x,t_1,\dots,t_k)\,\sigma_1(dt_1)\dots\sigma_k(dt_k),$$

where σ_i denotes the (α_i-1) — dimensional Lebesgue measure on ∂G_i, $i=1,\dots,k$;

$$W_\alpha(f,x_1,\dots,x_k,t_1,\dots,t_k)$$

$$=\sum_{I,J}(-1)^p\,\frac{\partial^r E(x_1-t_1,\dots,x_k-t_k)}{\partial\vec{n}_{t_{j_1}},\dots,\partial\vec{n}_{t_{j_r}}}\;\frac{\partial^p f(t_1,\dots,t_k)}{\partial\vec{n}_{t_{i_1}},\dots,\partial\vec{n}_{t_{i_p}}},$$

where $I=(i_1,\dots,i_r)$, $J=(j_1,\dots,j_p)$, $I\cap J=\phi$, $p+r=k$, $\{\vec{n}_{t_i}\}_{t_i\in\partial G_i}$ denotes the field of exterior normal vectors to ∂G_i, $i=1,\dots,k$.

Having this representation, in the proof that every α-polyharmonic function on D may be continued to arbitrarily continuable analytic (or, in the case $\alpha_i=2p_i\geqslant 4$, $i=1,\dots,k$, to holomorphic) function on $\tilde{D}_1\times\dots\times\tilde{D}_k$, we can apply (with only formal changes) the method of [4]. Hence $\tilde{D}_1\times\dots\times\tilde{D}_k$ satisfies $(A2)$ (or — $(H2)$).

Let $f_i\in H(D_i)$ satisfy $(A3)$ for $S_i=H(D_i)$, $i=1,\dots,k$. Then the function $f(x)=f_1(x_1),\dots,f_k(x_k)$, $x=(x_1,\dots,x_k)\in D$, is α-polyharmonic on D. Let $F_i\subset\mathcal{O}_{\alpha_i}$ be an arbitrarily continuable continuation of f_i over $\tilde{D}_i$, $i=1,\dots,k$; let $z=(z_1,\dots,z_k)\in\tilde{D}_1\times\dots\times\tilde{D}_k$, $(F_i)_{z_i}\in\pi_{\alpha_i}^{-1}(z_i)\cap F_i$, $\varphi_i\in(F_i)_{z_i}$, $\varphi_i\in\mathcal{O}(U_i)$, $z_i\in U_i=U_i^0\subset\tilde{D}_i$, $i=1,\dots,k$. Set $\varphi(w)=\varphi_1(w_1)\dots\varphi_k(w_k)$, $w=(w_1,\dots,w_k)\in U=U_1\times\dots\times U_k$; $\varphi\in\mathcal{O}(U)$. We take the germ φ_w of φ at w. Now we change $w\in U$, $(F_i)_{z_i}\in\pi_{\alpha_i}^{-1}(z_i)\cap F_i$ and $z\in\tilde{D}_1\times\dots\times\tilde{D}_k$.