ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCCXLI PRACE MATEMATYCZNE, ZESZYT 18 1977

Jan Malczak

On the Existence of Periodic Solutions for an Ordinary Differential Equation of the Second Order with a Non-differentiable Right Hand Side

The purpose of this note is to show that using the concept of multivalued differential we can prove existence theorems for periodic solutions and solutions of some boundary value problems for ordinary differential equations. Our main tool will be an implicit function theorem for non-differentiable mappings in Banach spaces. This theorem will give us a family of solutions which have the property of continuous dependence.

2. DEFINITIONS AND NOTATIONS

Let $(E, |\cdot|)$ be a Banach space. Let $cf(E)$ denote the set of all nonempty closed convex subsets of E. Throughout this section, U will denote an open subset of E. Let $p\colon U\to E$ and let $F\colon E\to cf(E)$ be homogeneous, i.e., for every $\lambda\in R$, $F(\lambda x)=\lambda F(x)$ for all $x\in E$.

Now let V be a normed space. We denote the ball in V with radius ε and center 0 by $B(\varepsilon)$. Let $x_0\in U$.

Definition 1. A homogeneous map $F\colon E\to cf(E)$ is called an upper differential of p at the point x_0, if there exists $\delta>0$ such that

$$p(x)-p(x_0)\in F(x-x_0)$$

for all x: $|x-x_0|<\delta$.

Definition 2. A map $D_p\colon E\to cf(E)$ is called a completely continuous differential of p at x_0, if

$$D_p(x) = \bigcap \left\{\begin{matrix}F(x)\text{: where } F \text{ is an upper differential of } p \text{ at } x_0 \text{ and } F \text{ is completely}\\ \text{continuous.}\end{matrix}\right\}$$

Using similar methods as in [6] it may easily be shown that:

Lemma 1. The function D_p: $E \to cf(E)$ is homogeneous and completely continuous. Furthermore, there exists a sequence $\{F_n\}$ of upper differentials such that

i) $F_{n+1}(x) \subset F_n(x)$ for all $x \in E$ and $n = 1, 2, \ldots$

ii) $D_p(x) = \bigcap_{n=1}^{\infty} F_n(x)$.

We now prove:

Lemma 2. Let the function D_p: $E \to cf(E)$ be as in Lemma 1 and the following implication be true

$$x \in D_p(x) \Rightarrow x = 0 .$$

Then, there exists $n_0 \geqslant 1$ such that

$$x \in F_{n0}(x) \Rightarrow x = 0 ,$$

where F_{n0} is the function from the conclusion of Lemma 1.

Proof. For if not, there exists a sequence $\{x_n\}$ satisfying $x_n \in F_n(x_n)$ and $x_n \neq 0$. Since F_n is homogeneous we have

$$\frac{x_n}{|x_n|} \in F_n\left(\frac{x_n}{|x_n|}\right).$$

Thus, since F_1 is completely continuous and $F_{n+1}(x) \in F_n(x)$ there exists a subsequence of $\{x_n\}$, which we call by x_n again, such that

$$\frac{x_n}{|x_n|} \to z, \ |z| = 1 .$$

But, by the upper semicontinuity of all F_n we have

$$z \in \bigcap_{n=1}^{\infty} F_n(z) = D_p(z) \Rightarrow z = 0 .$$

This completes the proof.

The following lemma is in [2]:

Lemma 3. Let F: $E \to cf(E)$ be a completely continuous and homogeneous function such that

$$x \in F(x) \Rightarrow x = 0 .$$

Then, there exists $J > 0$ such that

$$x \in F(x) + b \Rightarrow |x| \leqslant J|b| \quad \text{for} \quad x \in E .$$

From Lemma 2 and Lemma 3 we get

Corollary 1. Under the conditions of Lemma 2, there exists $J > 0$ such that

$$x \in F_{n0}(x) + b \Rightarrow |x| \leqslant J|b| .$$

Now we are able to state the following theorem

Theorem 1. Let

$$x = p(x)+q(x, v) \tag{1}$$

be a functional equation, where $p\colon U\to E$ and $q\colon U\times B(\alpha)\to E$ are completely continuous, $\alpha>0$, and $q(x, v)\to 0$ uniformly in x as $|v|\to 0$.

Suppose there exists $x_0 \in U$, such that $x_0 = p(x_0)$ and there exists the multivalued differential D_p of p at x_0. Assume the implication

$$x \in D_p(x) \Rightarrow x = 0\,.$$

Then there exists $\bar{\varepsilon} \in (0, \alpha)$ and $K>0$, such that
1° for every $v \in B(\bar{\varepsilon})$, there is a solution of (1) which satisfies $|x_v - x_0| < K$, and
2° $|x_v - x_0| \to 0$ as $|v|\to 0$.

The proof follows from Lemma 2 and Theorem 1 in [2].

III. BOUNDARY VALUE PROBLEM

Let $I = [0, 1]$, and C^n be the space of all continuous functions $x\colon I\to R^n$ with the usual norm topology, $|x| = \sup\{|x(t)|\colon t \in I\}$.

Consider a boundary value problem

$$\dot{x}(t) = f(t, x(t))\,, \quad L_0 x = r_0\,, \tag{2}$$

and a family of boundary value problems

$$\dot{x}(t) = f(t, x(t))+g(t, x(t), \varepsilon)\,, \quad L_\varepsilon x = r_\varepsilon\,, \tag{3}$$

where ε denotes a real parameter, and for each ε, L_ε maps C^n into R^n.

Using the generalized Fredholm Theorem (see [1]) and the above Theorem 1 we get immediately the following:

Theorem 2. Assume that:
1° F; $I\times R^n \to cf(R^n)$ is a function satisfying Carathéodory's condition.
2° $F(t, \cdot)$ is homogeneous.
3° The scalar function

$$\varphi(t) = \sup_{|p|\leqslant 1} \|F(t, p)\| \quad \text{where} \quad \|A\| = \sup_{q\in A} |q|$$

is integrable.
4° $f\colon I\times R^n \to R^n$ satisfy Carathéodory's condition.
5° $f(t, p)-f(t, p') \in F(t, p-p')$ for all $\begin{matrix}(t, p) \in I\times R^n \\ (t, p') \in I\times R^n\end{matrix}$

and

$$\int_0^1 |f(t, 0)|\,dt < +\infty\,.$$

6° L_ε: $C^n(I)\to R^n$ is linear and continuous for $|\varepsilon|<\varepsilon_0$.
7° the map: $\varepsilon\to L_\varepsilon$ is continuous.
8° g: $I\times R^n\times R\to R^n$ is continuous and such that $g(t,x,0)\equiv 0$.

Assume moreover that the boundary value problem

$$\dot{x}(t)\in F(t,x(t)),\quad L_0x=0$$

admits only a zero solution. Then

i) there exists exactly one solution x_0 of the boundary value problem (2),

ii) there exists $\bar{\varepsilon}>0$ such that for each ε with $|\varepsilon|<\bar{\varepsilon}$ there is a solution $x_\varepsilon(t)$ of the boundary value problem (3) and $x_\varepsilon(t)\to x_0(t)$ uniformly in t as $\varepsilon\to 0$.

Proof. We note that if p: $C^n(I)\times R^n\to C^n(I)\times R^n$ and $p(x,h)=(\bar{y},\bar{a})$ where

$$(\bar{y}(x))(t)=x(0)+\int_0^t f(s,x(s))ds+h,\ \bar{a}=L_0x-r_0+h$$

and

$$q\colon C^n(I)\times R^n\times[-\beta,\beta]\to C^n(I)\times R^n,\ q(x,h,\varepsilon)=(z,c),$$

where

$$z(t)=\int_0^t g(s,x(s),\varepsilon)ds-h,\ c=L_\varepsilon x-r_\varepsilon-(L_0x-r_0)$$

the problems (2) and (3) are equivalent to equation (1) when $\varepsilon=0$ and $\varepsilon\neq 0$ respectively. By using the techniques in [5], it can be observed that conditions in Theorem 1 on p and q are satisfied. The multivalued mapping P: $C^n(I)\times R^n\to cf(C^n(I)\times R^n)$, where $P(x,h)=(y,a)$ and

$$y(t)=\{x(0)+\int_0^t u(s)ds+h\colon u(s)\in F(s,x(s))\},$$
$$a=h+L_0x$$

can also be shown by the methods in [6], to satisfy the conditions in Theorem 1. With these observations, the conclusions of the Theorem 2 follow from Theorem 1.

IV. SOME APPLICATIONS

We are going to apply Theorem 2 to the problem of the existence and continuous dependence of periodic solutions of an ordinary differential equation of the second order.

Let us consider the following boundary value problem

(4) $$\ddot{x}=-\varphi(t,x,\dot{x},\varepsilon)\ \text{and}\ x(0)=x(\omega),\ \dot{x}(0)=\dot{x}(\omega)$$

where φ is periodic in t with a period ω.

Next, we consider the linear homogeneous equation

(5) $$\ddot{x}+p(t)x=0.$$

The following Lemma is due to A. Lasota and Z. Opial [3].

Lemma 4. If a function $p(t)$ is periodic with period ω, integrable on $[0, \omega]$, and the following conditions are satisfied

$$(\alpha) \qquad \int_0^{\omega} p(t)\,dt \geqslant 0, \quad \omega \int_0^{\omega} |p(t)|\,dt \leqslant 16, \quad p(t) \not\equiv 0 .$$

Then $x \equiv 0$ is the only periodic solution of (5) with period ω. We shall prove the following:

Theorem 3. Let $\varphi(t, x, y, \varepsilon)$ be a continuous function for $(t, x, y, \varepsilon) \in R^3 \times [-\beta, \beta]$ and periodic in t with period ω.

Let us assume that

1° $\varphi(t, x, y, 0)$ is constant with respect to y.

2° $p(t) \leqslant \dfrac{\varphi(t, x, y, 0) - \varphi(t, \bar{x}, y, 0)}{x - \bar{x}} \leqslant P(t),\ x > \bar{x}$

where $p(t)$ and $P(t)$ are periodic functions with period ω, integrable on $[0, \omega]$ such that

$$(\gamma) \qquad p(t) \not\equiv 0, \quad \int_0^{\omega} p(t)\,dt \geqslant 0, \quad \omega \int_0^{\omega} P(t)\,dt \leqslant 16$$

Then

a) there exists exactly one periodic solution $x_0(t)$ of equation

$$\ddot{x} = -\varphi(t, x, \dot{x}, 0)$$

with period ω.

b) there exists $\bar{\varepsilon} \in (0, \beta)$ and $K > 0$ such that for each ε with $|\varepsilon| < \bar{\varepsilon}$ there is a periodic solution $x_\varepsilon(t)$ with period ω of problem (4) such that

$$|x_\varepsilon(t) - x_0(t)| < K$$

c) $x_\varepsilon(t) \to x_0(t)$ uniformly with respect to t as $\varepsilon \to 0$.

Proof. Define the multivalued function F by the formula

$$F(t, x, y) = \{(y, -\lambda x) \colon p(t) \leqslant \lambda \leqslant P(t)\} .$$

From Lemma 4 it follows that the problem

$$(\dot{x}, \dot{y}) \in F(t, x, y), \quad x(0) = x(\omega), \quad y(0) = y(\omega) .$$

admits only a solution identically equal to zero.

Observe that $F(t, x, y)$ satisfies all conditions of Theorem 2, i.e., $F(t, x, y)$ satisfies Carathéodory's condition, is homogeneous with respect to (x, y) and the scalar function

$$\begin{aligned} \psi(t) &= \sup\{|F(t, x, y)| \colon x^2 + y^2 = 1\} \\ &= \sup\{\sqrt{x^2 + \lambda^2 \cdot y^2} \colon x^2 + y^2 = 1, \quad p(t) \leqslant \lambda \leqslant P(t)\} , \end{aligned}$$

is integrable on $[0, \omega]$.

Now writing the problem (4) in the form

$$\begin{aligned} \dot{x} &= y & x(0) &= x(\omega) \\ \dot{y} &= -\varphi(t, x, y, \varepsilon), & y(0) &= y(\omega) . \end{aligned}$$

or equivalently

$$\begin{aligned} \dot{x} &= y & x(0) &= x(\omega) \\ \dot{y} &= -\varphi(t, x, y, 0) + [\varphi(t, x, y, 0) - \varphi(t, x, y, \varepsilon)], & y(0) &= y(\omega) \end{aligned}$$

and putting

$$\begin{aligned} f(t, x, y) &= (y, -\varphi(t, x, y, 0)) \\ g(t, x, y, \varepsilon) &= (0, \varphi(t, x, y, 0) - \varphi(t, x, y, \varepsilon)) \end{aligned}$$

we get

$$f(t, x, y) - f(t, \hat{x}, \hat{y}) \in F(t, x - \hat{x}, y - \hat{y}) \quad \text{for all} \quad \begin{array}{l} (t, x, y) \in R^3 \\ (t, \hat{x}, \hat{y}) \in R^3 . \end{array}$$

Thus, since the functions $f(t, x, y)$ and $g(t, x, y, \varepsilon)$ satisfy all assumptions of Theorem 2 we obtain the desired result. The proof is complete.

REFERENCES

[1] A. Lasota, *Une généralisation du premier theoréme de Fredholm et ses applications à la theorie des équations différentielles ordinaires*, Ann. Polon. Math. 18 (1966).

[2] S. N. Chow and A. Lasota, *An implicit function theorem for non-differentiable mappings*. Proc. Amer. Math. Soc. Vol. 34. No. 1. 1972.

[3] A. Lasota et Z. Opial, *Sur les solutions périodiques des équations différentielles ordinaires*. Ann. Polon. Math. 16 (1964), p. 69—94.

[4] M. A. Krasnoselskij, *Topological methods in the theory of nonlinear integral equations* GITTL, Moscow, 1956, English trans. Macmillan, New York, 1964.

[5] A. Lasota and Z. Opial, *An application of the Kakutani — Ky Fan Theorem in the theory of ordinary differential equations*. Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. 13 (1965), 781—786.

[6] J. Malczak, *Asymptotic behavior for nonautonomous differential equation with multivalued differential* (in press).