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Asymptotic Behaviour for a Nonautonomous Differential Equation with Multivalued Differential

I. INTRODUCTION

For ordinary differential equations the following result is well-known [1].

Theorem 1. Let $f: R^{n+1} \rightarrow R^n$ be a continuous function with respect to $(t, x) \in R^{n+1}$ and continuously differentiable with respect to $x = (x_1, \dots, x_n)$. Furthermore, let $f(t+1, x) = f(t, x)$ for all $(t, x) \in R^{n+1}$.

Let us denote

$$f_x(t, x) = \left(\frac{\partial f_i}{\partial x_j}(t, x) \right)_{i, j = 1, \dots, n}$$

If all solutions $u(\cdot)$ of the linear system

$$(1) \quad \dot{u} = f_x(t, 0)u$$

satisfy $u(t) \rightarrow 0$ as $t \rightarrow \infty$, then there exist positive constants δ , K and σ such that all solutions $x(\cdot)$ of

$$(2) \quad \dot{x} = f(t, x)$$

with $|x(0)| < \delta$ satisfy $|x(t)| \leq K|x(0)|e^{-\sigma t}$ for $t \geq 0$.

The main purpose of this paper is to extend results of this type to the case in which f is not necessarily continuously differentiable. Such an f may not possess a Frechet differential but, rather, a "multivalued differential" D_f of f with respect to x .

This will be, roughly speaking, the closed convex hull of all possible differential quotients of f in the direction of x .

We shall replace (1) by a corresponding differential equation of multivalued type

$$(3) \quad \dot{u} \in D_f(t, u)$$

We shall show that (3) approximates (2) in much the same way as (1) approximates (2).

Before the proof of our main result Theorem 6 we shall give a precise definition of the "multivalued differential" of f . Next, we shall establish some results which may be of some interest in themselves, mainly Lemma 6 and Theorems 2, 3, 4, 5 and which provide the mathematical basis for the proof of Theorem 6.

The Lemmas 1, 4, 6 and the Theorems 2, 3, 4, 5, 6 which will be presented in this paper are an extension of the corresponding results for the autonomous differential equations presented in [2].

II

Let R^n denote a Euclidean n -dimensional space. For $x \in R^n$ define $|x| = |x_1| + \dots + |x_n|$. Let $A \subset R^n$, $B \subset R^n$ and $r > 0$. We define

$$d(x, A) = \inf\{|x-y|: y \in A\}$$

$$|A| = \sup\{|x|: x \in A\}$$

$$A+B = \{a+b: a \in A, b \in B\}$$

$$N(A, r) = \{x \in R^n: d(x, A) < r\}$$

$$B(r) = \{x \in R^n: |x| \leq r\}$$

Let C^n denote the set of all nonempty compact convex subsets of R^n . We recall the notion of the upper semicontinuousness of a multivalued map.

Definition 1. Let $F: R^{n+1} \rightarrow C^n$. We say that F is upper semicontinuous (u.s.c) if, for each $(t, x) \in R^{n+1}$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(s, y) \subset N(F(t, x), \varepsilon) \text{ whenever } |(s, y) - (t, x)| < \delta$$

Remark 1. It is shown in [3] that if F maps R^{n+1} into C^n , and for each $(t, x) \in R^{n+1}$, there exists a neighbourhood $N(t, x)$ such that $\bigcup_{(s, y) \in N} F(s, y)$ is bounded, then F is u.s.c. if and only if the graph of F is closed in $R^{n+1} \times R^n$.

Definition 2. We say that $f: R^{n+1} \rightarrow R^n$ is Lipschitzian at $(t, 0)$ if there exist a function: $t \rightarrow L(t)$ and a number $\eta > 0$ such that

$$|f(t, x) - f(t, 0)| \leq L(t)|x| \text{ for all } |x| \leq \eta$$

Definition 3. Let $f: R^{n+1} \rightarrow R^n$ be given. A function $F: R^{n+1} \rightarrow C^n$ is called an upper differential of f at $(t, 0)$ if:

- i) F is upper semicontinuous.
- ii) F is homogeneous in x , i.e., $F(t, \lambda x) = \lambda F(t, x)$ for all $x \in R^n$, $\lambda > 0$.
- iii) there exists $\delta > 0$ such that

$$f(t, x) - f(t, 0) \in F(t, x) \text{ for all } |x| \leq \delta.$$

Remark 2. If in definition 2 we suppose that $L(\cdot)$ is u.s.c. then for an f there always exists an upper differential. For example the multivalued function $F(t, x) = L(t)B(|x|)$ is this upper differential.

Definition 4. Let $f: R^{n+1} \rightarrow R^n$ be given. We define the multivalued differential D_f of f at $(t, 0)$ by:

$$D_f(t, x) = \bigcap \{F(t, x): F \text{ is an upper differential of } f \text{ at } (t, 0)\}.$$

Lemma 1. Let $f: R^{n+1} \rightarrow R^n$ be Lipschitzian at $(t, 0)$ with an u.s.c. coefficient $L(\cdot)$. Then there exists a homogeneous multivalued differential D_f which is u.s.c.

Furthermore, there exists a sequence $\{F_n\}$ of upper differentials such that

$$i) F_{n+1}(t, x) \subset F_n(t, x), (t, x) \in R^{n+1}, \quad n = 1, 2, \dots$$

$$ii) D_f(t, x) = \bigcap_{n=1}^{\infty} F_n(t, x).$$

Proof. It follows directly from the definition of D_f that $D_f(t, x)$ is compact, convex and that D_f is homogeneous with respect to x . We now show that D_f is nonempty for all $(t, x) \in R^{n+1}$ and upper semicontinuous.

Let Γ_{D_f} denote the graph of D_f , i.e.,

$$\Gamma_{D_f} = \{(t, x, y) : y \in D_f(t, x)\} \subset R^{n+1} \times R^n.$$

From the definition of D_f we have

$$(4) \quad \Gamma_{D_f} = \Gamma_{\cap F} = \bigcap \Gamma_F$$

where the intersection ranges over all the upper differentials F of f at $(t, 0)$. By the Lindelöf property there exists a sequence $\{\Gamma_{G_n}\}$ of the graphs of G_n such that

$$\Gamma_{D_f} = \bigcap_{n=1}^{\infty} \Gamma_{G_n}$$

This implies that

$$\Gamma_{D_f} = \Gamma_{\bigcap_{n=1}^{\infty} G_n}.$$

Thus

$$D_f(t, x) = \bigcap_{n=1}^{\infty} G_n(t, x).$$

Now we define

$$F_n(t, x) = \bigcap_{k=1}^n G_k(t, x).$$

Then F_n is also an upper differential and

$$D_f(t, x) = \bigcap_{n=1}^{\infty} F_n(t, x).$$

By virtue of Cantor's theorem the last intersection is nonempty. From Remark 1 and from (4) it follows that D_f is upper semicontinuous. The proof is complete.

Definition 5. Let $F: R^{n+1} \rightarrow C^n$ be u.s.c. A solution of the multivalued equation

$$(5) \quad \dot{x} \in F(t, x)$$

is defined to be an absolutely continuous function $x(\cdot)$ defined on a non-degenerated interval I such that

$$\dot{x}(t) \in F(t, x(t)) \text{ for almost all } t \text{ in } I.$$

The existence of solutions of (5) is discussed in [4] where is stated the following:

Lemma 2. Let $F: R^{n+1} \rightarrow C^n$ be u.s.c., $x_0 \in R^n$, $t_0 \in R$. Then there exist $\alpha > 0$ and at least one solution $x(\cdot)$ of (5) defined on $(t_0 - \alpha, t_0 + \alpha)$ satisfying $x(t_0) = x_0$.

Similarly, the following result is true:

Lemma 3. Let $F: R^{n+1} \rightarrow C^n$ be u.s.c. Let $x(\cdot)$ be a solution of $\dot{x} \in F(t, x)$ on an interval (S, T) . Then either

- i) there exists $\alpha > 0$ such that $x(\cdot)$ can be continued as a solution of $\dot{x} \in F(t, x)$ to $(S, T + \alpha)$, or
- ii) $|x(t)| \rightarrow \infty$ as $t \rightarrow T$.

The analogous result is true at the left endpoint S . Hence we have

Corollary 1. Let $F: R^{n+1} \rightarrow C^n$ be u.s.c. Let Q be a compact subset of R^n and let $\varphi(t)$ be continuous for all real t .

Suppose that every solution $x(\cdot)$ of

$$(6) \quad \dot{x} \in F(t, x), x(0) \in Q$$

satisfies $|x(t)| \leq \varphi(t)$ on its interval of definition.

Then every solution $x(\cdot)$ of (6) can be continued to all $(-\infty, +\infty)$ and each such continuation satisfies

$$|x(t)| \leq \varphi(t) \text{ for all real } t.$$

We prove the following:

Lemma 4. Let $F: R^{n+1} \rightarrow C^n$ be u.s.c. and $F(t, x) = F(t+1, x)$. Let $0 < K_1 < K_2$. Then there exists $\delta > 0$ such that every solution $x(\cdot)$ of

$$\dot{x} \in F(t, x) \quad \text{with} \quad |x(t_0)| \leq K_1$$

satisfies

$$|x(t)| \leq K_2 \quad \text{for all } |t - t_0| \leq \delta.$$

Proof. Let

$$M = \sup \{ \|F(t, x)\| : |x| \leq K_2, 0 \leq t \leq 1 \}.$$

By virtue of the integral representation of solution $x(\cdot)$ of $\dot{x} \in F(t, x)$ (see [5]) we obtain

$$x(t) \in x(t_0) + \int_{t_0}^t F(s, x(s)) ds$$

or

$$x(t) - x(t_0) \in \int_{t_0}^t F(s, x(s)) ds.$$

Thus

$$\|x(t) - x(t_0)\| \leq \|x(t) - x(t_0)\| \leq \left\| \int_{t_0}^t F(s, x(s)) ds \right\|.$$

1° Suppose that $x(t_0) = K_1$. Then since $|x(t) - K_1| \leq M|t - t_0|$ we obtain

$$M|t - t_0| \leq K_2 - K_1 \quad \text{for} \quad |t - t_0| \leq \frac{K_2 - K_1}{M}.$$

2° If $|x(t_0)| < K_1$, then since the function $x(\cdot)$ is continuous, there is $t_1 > t_0$ such that $x(t_1) = K_1$ and $|x(t)| < K_1$ for all $t \in (t_0, t_1)$.

Now repeating the argument of point 1 for t_1 we obtain the desired result. The proof is complete.

Our final preparatory lemma says that the limit of solutions of a multivalued equation is also a solution. Its proof may be found in [3].

Lemma 5. Let $F: R^{n+1} \rightarrow \mathcal{C}^n$ be u.s.c. and let $\{x_n(\cdot)\}$ be an infinite sequence of solutions of $\dot{x} \in F(t, x)$ defined on a finite interval I . Suppose $x_n(t) \rightarrow x(t)$ uniformly for all $t \in I$ and suppose that $x(\cdot)$ is absolutely continuous on I .

Then $x(\cdot)$ is a solution of $\dot{x} \in F(t, x)$.

III

Now we shall prove some results of the behaviour of solutions of multivalued differential equations.

Lemma 6. Let $F: R^{n+1} \rightarrow \mathcal{C}^n$ be u.s.c. and homogeneous in x . Suppose that F is periodic in t with period 1. Then there exists $\alpha > 0$ such that every solution $x(\cdot)$ of $\dot{x} \in F(t, x)$ exists on $(-\infty, +\infty)$ and satisfies

$$|x(t)| \leq |x(t_0)| e^{\alpha|t-t_0|} \quad \text{for all real } t \text{ and } t_0.$$

Proof. We first show that there exists $\alpha > 0$ such that

$$F(t, x) \subset \alpha B(|x|) \quad \text{for all } x \neq 0 \text{ and } 0 \leq t \leq 1,$$

where $B(|x|) = \{y \in R^n: |y| \leq |x|\}$.

Suppose that $|x| = 1$. Then there exists $\varrho = \varrho(t, x) > 0$ such that $F(t, x) \subset B(\varrho)$, since $F(t, x)$ is compact.

Next, since F is upper semicontinuous there exists $\delta = \delta(t, x) > 0$ such that $F(t, y) \subset B(\varrho + 1)$ for all $(t, y) \in N((t, x), \delta)$ and $|y| = 1$. As (t, x) ranges over the set $[0, 1] \times \{x: |x| = 1\}$; the neighbourhoods $N((t, x), \delta)$ covering the set $[0, 1] \times \{x: |x| = 1\}$. Thus there exists a finite subcovering $N((t_1, x_1), \delta_1), \dots, N((t_q, x_q), \delta_q)$.

Define

$$\alpha = \max\{\varrho(t_1, x_1) + 1, \dots, \varrho(t_q, x_q) + 1\}.$$

Then

$$F(t, x) \subset B(\alpha) = \alpha B(1) \quad \text{for all } |x| = 1, 0 \leq t \leq 1.$$

Now for any $x \neq 0$ we have

$$F(t, x) = |x| F\left(t, \frac{x}{|x|}\right) \subset |x| B(\alpha) = \alpha B(|x|)$$

which was to be shown.

If $x(\cdot)$ satisfies $\dot{x} \in F(t, x)$, then by the above argument, it also satisfies

$$|\dot{x}(t)| \leq \alpha |x(t)|.$$

Integrating this inequality and applying Corollary 2 we obtain the desired result.

Now we shall prove a perturbation theorem for multivalued differential equations.

Theorem 2. Let $F_n: R^{n+1} \rightarrow C^n$ be u.s.c. function for $n = 1, 2, \dots$. Assume that $F_{n+1}(t, x) \subset F_n(t, x)$ for all $(t, x) \in R^{n+1}$ and $n = 1, 2, \dots$. Let us define

$$F(t, x) = \bigcap_{n=1}^{\infty} F_n(t, x)$$

Let F_n be periodic in t with the period 1, and homogeneous in x .

Suppose that all solutions of $\dot{x} \in F(t, x)$ approach zero as $t \rightarrow \infty$. Then there exist a positive integer m and $K > 0$ such that every solution $x(\cdot)$ of

$$\dot{x} \in F_m(t, x), |x(0)| \leq 1$$

satisfies

$$|x(t)| \leq K \quad \text{for all } t \geq 0.$$

Proof. Arguing as in Lemma 1 we obtain that F is u.s.c. Thus it makes sense to speak of solutions of $\dot{x} \in F(t, x)$.

Suppose the conclusion of Theorem 2 is false. So, let t_n be the smallest value of $t \geq 0$ such that there is a solution of

$$(7) \quad \dot{x} \in F_n(t, x), |x(0)| \leq 1$$

satisfying $|x(t)| \geq n+1$. We call such a solution $x_n(\cdot)$. We claim that $|x_n(t)| > r = e^{-\frac{1}{2}\alpha_1}$ for all $0 < t < t_n$, where α_1 is a constant in the Lemma 6 corresponding to the function $F_1(t, x)$.

If not, there exists α satisfying $0 < \alpha < t_n$ such that $|x_n(\alpha)| \leq e^{-\frac{1}{2}\alpha_1} < 1$.

Now we consider two cases:

1° The number α is an integer. Define $y(t) = x_n(t+\alpha)$, which by virtue of the above inequality satisfies (7) and $|y(t_n-\alpha)| = |x_n(t_n-\alpha+\alpha)| = n+1$ which contradicts the definition of t_n .

2° There is an integer l such that $l < \alpha < l+1$. At least one of the intervals $[l, \alpha]$, $[\alpha, l+1]$ has a length not greater than $\frac{1}{2}$. Let it be the interval $[\alpha, l+1]$.

By Lemma 6 we obtain on this interval the estimation

$$|x(t)| \leq |x(\alpha)| e^{\alpha_1|t-\alpha|}$$

and at the point $l-1$ we have

$$|x(l-1)| \leq e^{-\frac{1}{2}\alpha_1} e^{\alpha_1|l+1-\alpha|} \leq e^{-\frac{1}{2}\alpha_1} e^{\frac{1}{2}\alpha_1} = 1.$$

Hence, arguing as in the case 1° we obtain a contradiction with the definition of t_n . This proves the claim. The solution $x_n(\cdot)$ of (7) is also a solution of

$$\dot{x} \in F_{n-1}(t, x), |x(0)| \leq 1.$$

Thus $|x_n(t)| \leq n$ for $0 \leq t \leq t_{n-1}$ by definition of t_{n-1} . Hence $t_{n-1} < t_n$. This means that

$$0 < t_1 < t_2 < \dots < t_n < \dots$$

Now there are two cases to be considered:

a) First, suppose $t_n \rightarrow \infty$. Then $|x_n(\cdot)| \leq 2$ on $[0, t_1]$ for $n = 1, 2, \dots$. Since each $x_n(\cdot)$ satisfies $\dot{x} \in F_1(t, x)$ we have $|\dot{x}_n(t)| \leq M_1$ for almost all t in $[0, t_1]$ where

$$M_1 = \sup\{|F_1(t, x)| : |x| \leq 2, 0 \leq t \leq 1\}.$$

Thus the family $x_n(\cdot)$ is equicontinuous and uniformly bounded on $[0, t_1]$.

By Ascoli's Lemma there is a continuous function $x(\cdot)$ and a subsequence $\{x_{n_1}(\cdot)\}$ of $\{x_n(\cdot)\}$ such that $x_{n_1}(t) \rightarrow x(t)$ uniformly for $0 \leq t \leq t_1$.

Thus $x(\cdot)$ is absolutely continuous on $[0, t_1]$. Similarly, each $x_{n_1}(\cdot)$ is bounded in norm by number 3 on $[0, t_2]$ and has the Lipschitz constant M_2 , where

$$M_2 = \sup\{|F_2(t, x)| : |x| \leq 3, 0 \leq t \leq 1\}.$$

Thus a subsequence $\{x_{n_2}(\cdot)\}$ of $\{x_{n_1}(\cdot)\}$ converges uniformly on $[0, t_2]$ to an absolutely continuous function. Since this limit function is equal to $x(\cdot)$ on $[0, t_1]$, it still may be denoted by $x(\cdot)$ on $[0, t_2]$.

We repeat this procedure for $[0, t_k]$, $k = 1, 2, \dots$ and we obtain the extension of the previous limit function.

Now we consider the diagonal subsequence $\{x_{n_i}(\cdot)\}$ and we denote this subsequence by $\{x_{n_i}(\cdot)\}$.

Thus the function $x(t)$ is defined on $[0, \infty)$ and such that

$$x_{n_i}(t) \rightarrow x(t) \text{ uniformly for } 0 \leq t \leq t_k.$$

Now let us fix k . Then $x_{n_i}(\cdot)$ satisfies $\dot{x} \in F_k(t, x)$ on $[0, t_k]$ for all sufficiently large i . By Lemma 5 also $x(\cdot)$ satisfies $\dot{x} \in F_k(t, x)$ on $[0, t_k]$. Since k was arbitrary $x(\cdot)$ satisfies

$$\dot{x} \in F(t, x), |x(0)| \leq 1 \text{ on } [0, \infty).$$

Thus, by hypothesis we obtain $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

On the other hand from the property $|x_{n_i}(t)| > N$ for all $0 < t < t_{n_i}$ it follows that $|x(t)| \geq r$ for all $t \geq 0$. This contradicts the assumptions of Theorem 2.

b) Suppose $\{t_n\}$ is bounded. Since $\{t_n\}$ is monotonic, there is some $T > 0$ such that $t_n \rightarrow T$. Arguing as in the previous case, we may prove the existence of a subsequence $\{x_{n_i}(\cdot)\}$ of $\{x_n(\cdot)\}$ which converges to $x(\cdot)$ uniformly on compact subsets of $[0, T)$, where $x(\cdot)$ is absolutely continuous and satisfies the condition

$$\dot{x} \in F(t, x), |x(0)| \leq 1.$$

Thus, by hypothesis, $x(\cdot)$ can be continued up to $[0, \infty)$ with $x(t) \rightarrow 0$ as $t \rightarrow \infty$. In particular, there exists $P > 0$ such that $|x(t)| \leq P$ for $0 \leq t \leq T$. Now all $x_{n_i}(\cdot)$ and $x(\cdot)$ satisfies $\dot{x} \in F_1(t, x)$. Let us choose $\delta > 0$ as in Lemma 4, corresponding to $\dot{x} \in F_1(t, x)$, $K_1 = P + 1$, $K_2 = P + 2$.

Let us choose N so large that $|t_N - T| < \delta$. Since $x_{n_i}(t_N) \rightarrow x(t_N)$ as $i \rightarrow \infty$, we have $|x_{n_i}(t_N)| \leq P + 1$ for a sufficiently large i .

Thus, by Lemma 4 we obtain

$$|x_{n_i}(t)| \leq P + 2 \text{ for all } |t - t_N| < \delta.$$

But this contradicts conditions $|x_{n_i}(t_{n_i})| \rightarrow \infty$ and $|t_{n_i} - t_N| \leq |T - t_N| < \delta$ for a sufficiently large i . So, the proof of Theorem 2 is complete.

Theorem 3. Let $F_n: R^{n+1} \rightarrow C^n$ be u.s.c. function for $n = 1, 2, 3, \dots$. Assume that $F_{n+1}(t, x) \subset F_n(t, x)$ for all $(t, x) \in R^{n+1}$ and $n = 1, 2, \dots$. Let us define

$$F(t, x) = \bigcap_{n=1}^{\infty} F_n(t, x).$$

Let F_n be periodic in t with the period 1, and homogeneous in x .

Suppose that all solutions of

$$\dot{x} \in F(t, x)$$

approach zero as $t \rightarrow \infty$. Then there exist number m and $K > 0$ such that every solution $x(\cdot)$ of

$$x \in F_m(t, x)$$

satisfies

$$|x(t)| \leq K|x(0)| \quad \text{for all } t \geq 0.$$

Proof. By Theorem 2, there exist m and $K > 0$ such that every solution $x(\cdot)$ of

$$\dot{x} \in F_m(t, x), \quad |x(0)| \leq 1$$

satisfies

$$|x(t)| \leq K \quad \text{for all } t \geq 0.$$

Now let $y(\cdot)$ be any solution of $\dot{x} \in F_m(t, x)$. Suppose first that $y(0) \neq 0$. Define

$$x(t) = y(t)|y(0)|^{-1}.$$

Since

$$\dot{y}(t) \in F_m(t, y(t)).$$

Thus

$$y(t)|y(0)|^{-1} \in |y(0)|^{-1} \cdot F_m(t, y(t)).$$

Hence

$$\dot{x}(t) \in F_m(t, x(t)).$$

But

$$|x(0)| = |y(0)||y(0)|^{-1} = 1. \quad \text{Thus} \\ |x(t)| \leq K \quad \text{for all } t \geq 0.$$

Thus

$$|y(t)| \leq K|y(0)| \quad \text{for all } t \geq 0.$$

Now let $y(0) = 0$. We claim $y(t) = 0$ for all $t \geq 0$. Suppose not. Define

$$\tau = \sup\{t: y(s) = 0 \text{ for } 0 \leq s \leq t\}$$

Choose $\alpha > 0$ and $\varepsilon > 0$. There exists β , $\tau < \beta < \tau + \alpha$, such that $0 < |y(\beta)| < \varepsilon K^{-1}$ (by the definition of τ and by the continuity of $y(\cdot)$). As in the previous case we may prove that

$$|y(t)| \leq K|y(\beta)| \quad \text{for } t \geq \beta.$$

