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## Ultraproducts of Higher-Order Models and Non-standard Analysis

In [1] A. Robinson has stated the existence of a certain proper enlargement  $R^*$  of the model of analysis. In this paper we give an effective construction of  $R^*$  introducing the notion of the ultraproduct of higher-order models.

We define by induction the set  $T$  in the following way:

- 1)  $1 \in T$ ,
- 2) if  $\tau_1, \dots, \tau_n \in T$  then  $(\tau_1, \dots, \tau_n) \in T$ .

Elements of  $T$  will be called types. For every type  $\tau$  we define the rank of  $\tau$  ( $r(\tau)$ ) by induction:

- 1)  $r(1) = 0$ ,
- 2)  $r((\tau_1, \dots, \tau_n)) = 1 + \max_{i=1, \dots, n} r(\tau_i)$ .

For every set  $A$  and type  $\tau$  we put:

- 1)  $\tau A = A$  if  $\tau = 1$ ,
- 2)  $\tau A = P(\tau_1 A \times \dots \times \tau_n A)$  if  $\tau = (\tau_1, \dots, \tau_n)$  where  $P(X) = \{Y: Y \subset X\}$ . For a given mapping  $f: A \rightarrow B$  we define the mapping  $\tau f: \tau A \rightarrow \tau B$  in the following way:

- 1)  $1f = f$ ,
- 2) for  $C \in \tau A$ ,  $C \subset \tau_1 A \times \dots \times \tau_n A$  ( $\tau = (\tau_1, \dots, \tau_n)$ ) we put

$$\tau f(C) = \{(\tau_1 f(c_1), \dots, \tau_n f(c_n)): (c_1, \dots, c_n) \in C\}$$

where  $\tau_i f: \tau_i A \rightarrow \tau_i B$  ( $i = 1, \dots, n$ ) are already defined. Obviously  $\tau f(C) \in \tau B$ . It is easy to see that:

- 1)  $\tau(f \circ g) = \tau f \circ \tau g$  whenever  $f \circ g$  exists,
- 2)  $\tau(id_A) = id_{\tau A}$ ,
- 3) whenever  $f$  is an injection (surjection), so is  $\tau f$ .

We introduce the higher-order language  $L$ . The atomic symbols of  $L$  are:

- 1) variables  $(v, v_1, v_2, \dots)$  of type  $\tau$  for each  $\tau \in T$ .  $V^\tau$  is the set of all variables of type  $\tau$ .  $V$  is the set of all variables of  $L$ .
- 2) constants  $(\alpha, \beta, \gamma, \dots)$  of type  $\tau$  for each  $\tau \in T$ . In a similar way we introduce the sets  $R^*$ ,  $R$ .

