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Ultraproducts of Higher-Order Models and Non-standard Analysis

In [1] A. Robinson has stated the existence of a certain proper enlargement R^* of the model of analysis. In this paper we give an effective construction of R^* introducing the notion of the ultraproduct of higher-order models.

We define by induction the set T in the following way:

- 1) $1 \in T$,
- 2) if $\tau_1, \dots, \tau_n \in T$ then $(\tau_1, \dots, \tau_n) \in T$.

Elements of T will be called types. For every type τ we define the rank of τ ($r(\tau)$) by induction:

- 1) $r(1) = 0$,
- 2) $r((\tau_1, \dots, \tau_n)) = 1 + \max_{i=1, \dots, n} r(\tau_i)$.

For every set A and type τ we put:

- 1) $\tau A = A$ if $\tau = 1$,
- 2) $\tau A = P(\tau_1 A \times \dots \times \tau_n A)$ if $\tau = (\tau_1, \dots, \tau_n)$ where $P(X) = \{Y: Y \subset X\}$. For a given mapping $f: A \rightarrow B$ we define the mapping $\tau f: \tau A \rightarrow \tau B$ in the following way:

- 1) $1f = f$,
- 2) for $C \in \tau A$, $C \subset \tau_1 A \times \dots \times \tau_n A$ ($\tau = (\tau_1, \dots, \tau_n)$) we put

$$\tau f(C) = \{(\tau_1 f(c_1), \dots, \tau_n f(c_n)): (c_1, \dots, c_n) \in C\}$$

where $\tau_i f: \tau_i A \rightarrow \tau_i B$ ($i = 1, \dots, n$) are already defined. Obviously $\tau f(C) \in \tau B$. It is easy to see that:

- 1) $\tau(f \circ g) = \tau f \circ \tau g$ whenever $f \circ g$ exists,
- 2) $\tau(id_A) = id_{\tau A}$,
- 3) whenever f is an injection (surjection), so is τf .

We introduce the higher-order language L . The atomic symbols of L are:

- 1) variables (v, v_1, v_2, \dots) of type τ for each $\tau \in T$. V^τ is the set of all variables of type τ . V is the set of all variables of L .
- 2) constants $(\alpha, \beta, \gamma, \dots)$ of type τ for each $\tau \in T$. In a similar way we introduce the sets R^* , R .

3) the connectives \neg, \wedge ;

4) the quantifier \exists .

In any given situation we shall suppose that a sufficient supply of variables and constants is available in our language. The atomic formulas of L are of the form $s(t_1, \dots, t_n)$ where $s \in V^r \cup R^r$, $t_i \in V^{r_i} \cup R^{r_i}$ ($i = 1, \dots, n$) and $\tau = (\tau_1, \dots, \tau_n)$. The set of higher-order formulas of L is the smallest set \mathcal{F} such that

1) \mathcal{F} contains all atomic formulas of L ,

2) \mathcal{F} is closed under the application of connectives,

3) $\exists v F$ is in \mathcal{F} whenever F is in \mathcal{F} , $v \in V$ and v is free in F .

The pair $M = (\underline{M}, \psi_M)$ will be called a higher-order model if \underline{M} is any set and ψ_M is a partial mapping from R into $\bigcup_{\tau \in T} \tau \underline{M}$ such that for all $\tau \in T$ ψ_M maps R^r into $\tau \underline{M}$. We denote

$\text{dom } \psi_M \cap R^r = \overline{M}^r$, $\overline{M} = \bigcup_{\tau \in T} \overline{M}^r$, $\tau \underline{M} = \{\psi_M(\alpha) : \alpha \in \overline{M}^r\}$. We assume $\overline{M}^r \neq \emptyset$ for all

$\tau \in T$. A model $M = (\underline{M}, \psi_M)$ will be called full (normal) iff for all $\tau \in T$ $\psi_M: \overline{M}^r \rightarrow \tau \underline{M}$ is a surjection (injection). A model $M = (\underline{M}, \psi_M)$ will be called regular iff

1) $\underline{M} \neq \emptyset$,

2) $1 \underline{M} = \underline{M}$,

3) for all $\tau \neq 1$, $\tau = (\tau_1, \dots, \tau_n)$ we have

$$\tau \underline{M} \subset P(\tau_1 \underline{M} \times, \dots, \times \tau_n \underline{M})$$

This means if $(a_1, \dots, a_n) \in \psi_M(\alpha)$, $\alpha \in \overline{M}^r$, then there exist $\alpha_i \in \overline{M}^{r_i}$ such that $a_i = \psi_M(\alpha_i)$ ($i = 1, \dots, n$). We consider only regular models. The word "model" used in this paper will denote a regular one. For $F \in \mathcal{F}$, $\text{Fr} F$ denotes the set of all free variables of F and $\text{Fr}^r F = \text{Fr} F \cap V^r$. $\text{Val}_M F$ denotes the set of all valuations of formula F in model M i.e. mappings $\varphi: \text{Fr} F \rightarrow \bigcup_{\tau \in T} \tau \underline{M}$ such that $\text{im } \varphi|_{\text{Fr}^r F} \subset \tau \underline{M}$. For $t \in \text{Fr} F \cup \overline{M}$ we put

$\varphi_M t = \varphi(t)$ whenever $t \in \text{Fr} F$ and $\varphi_M t = \psi_M(t)$ in another case.

We say that formula F is satisfied in model M by valuation $\varphi \in \text{Val}_M F$ (shortly $M \models_\varphi F$) iff

1) All constants of F belongs to \overline{M} ,

2) a) $(\varphi_M t_1, \dots, \varphi_M t_n) \in \varphi_M s$ for atomic formula $F = s(t_1, \dots, t_n)$,

b) $M \not\models_\varphi H$ does not hold for F of the form $F = \neg H$,

c) $M \models_{\varphi|_{\text{Fr} F_1}} F_1$ and $M \models_{\varphi|_{\text{Fr} F_2}} F_2$ for $F = F_1 \wedge F_2$,

d) for some $a \in \tau \underline{M}$, $M \models_{\varphi \cup \{(v, a)\}} H$ for F of the form $F = \exists v H$ where $v \in \text{Fr}^r H$.

S denotes the set of all sentences of L . For $P \in S$, $\text{Val}_M P = \{\emptyset\}$ and we write $M \models P$ if $M \models_\emptyset P$. Models M, N will be called elementarily equivalent ($M \equiv N$) if for every $P \in S$ such that constants of P belong to $\overline{M} \cap \overline{N}$ we have: $M \models P$ iff $N \models P$. We call two models M, N isomorphic if there exist bijections $h: \underline{M} \rightarrow \underline{N}$, $h_r: \overline{M}^r \rightarrow \overline{N}^r$ (for each $\tau \in T$) such that for every $\alpha \in \overline{M}^r$ we have $\tau h(\psi_M(\alpha)) = \psi_N(h_r(\alpha))$. It is easy to see that if M, N are isomorphic, then

$$M \models P \quad \text{iff} \quad N \models P(\alpha_1/h_{r_1}(\alpha_1), \dots, \alpha_n/h_{r_n}(\alpha_n))$$

where $\{\alpha_1, \dots, \alpha_n\}$ are all constants of P and $\alpha_i \in \overline{M}^{r_i}$, ($i = 1, \dots, n$).

Let I be an arbitrary fixed index set and U an ultrafilter on I ($U \subset P(I)$). For each $i \in I$ let $M_i = (\underline{M}_i, \psi_i)$ be some model. We extend the set of constants of L . For every $\tau \in T$ we take $\overline{M}^\tau = \overline{M}^\tau \cup \prod_{i \in I} \overline{M}_i^\tau / U$ (see [2], 87). The ultraproduct of models M_i is the model $M = (\underline{M}, \psi)$ where $\underline{M} = \prod_{i \in I} \underline{M}_i / U$, $\overline{M}^\tau = \prod_{i \in I} \overline{M}_i^\tau / U$, and $\Psi: \overline{M}^\tau \rightarrow \tau \underline{M}$ is defined as follows: we take any $\bar{\alpha} \in \overline{M}^\tau$. $\bar{\alpha} = [\alpha]$, where $\alpha \in \prod_{i \in I} \overline{M}_i^\tau$;

- 1) If $\tau = 1$ we put $\psi(\bar{\alpha}) = [f]$ where $f(i) = \psi_i(\alpha(i))$.
- 2) If $\tau = (\tau_1, \dots, \tau_n)$ we say that $(b_1, \dots, b_n) \in \psi(\bar{\alpha})$ iff there exist $\bar{\alpha}_i = [\alpha_i] \in \overline{M}^{\tau_i}$ ($i = 1, \dots, n$) such that
 - a) $b_i = \psi(\bar{\alpha}_i)$,
 - b) $Z = \{i: (\psi_i(\alpha_1(i)), \dots, \psi_i(\alpha_n(i))) \in \psi_i(\alpha(i))\} \in U$.

It is obvious from the construction that M is regular. To verify the independence of the definition from the choice of representants from the equivalence classes $\bar{\alpha}$, $\bar{\alpha}_i$ we take any $\alpha' \in \bar{\alpha} = [\alpha]$, $\alpha'_i \in \bar{\alpha}_i = [\alpha_i]$.

$$Z_0 = \{i: \alpha'(i) = \alpha(i)\} \in U$$

and each

$$Z_i = \{i: \alpha'_i(i) = \alpha_i(i)\} \in U$$

Then

$$Z \cap Z_0 \cap \bigcap_{i=1}^n Z_i \subset W = \{i: (\psi_i(\alpha'_1(i)), \dots, \psi_i(\alpha'_n(i))) \in \psi_i(\alpha'(i))\}$$

hence $W \in U$. The independence from the choice of $\bar{\alpha}_i$ such that $b_i = \psi(\bar{\alpha}_i)$ will be the result of Theorem 1.

Theorem 1. Let $M = (\underline{M}, \psi)$ be some ultraproduct of models $\{M_i\}$. We have

- 1) $\psi(\bar{\alpha}) = \psi(\bar{\beta})$ iff $\{i: \psi_i(\alpha(i)) = \psi_i(\beta(i))\} \in U$ where $\bar{\alpha}, \bar{\beta} \in \overline{M}^\tau$; $\tau \in T$; $\bar{\alpha} = [\alpha]$, $\bar{\beta} = [\beta]$.
- 2) For every $\bar{\alpha} \in \overline{M}^\tau$, $\tau \neq 1$ ($\tau = (\tau_1, \dots, \tau_n)$) we have $\psi(\bar{\alpha}) = \{(a_1, \dots, a_n):$ for every $\bar{\beta}_1, \dots, \bar{\beta}_n$ such that $a_i = \psi(\bar{\beta}_i)$, $\bar{\beta}_i \in \overline{M}^{\tau_i}$ ($i = 1, \dots, n$) we have

$$\{i: (\psi_i(\beta_1(i)), \dots, \psi_i(\beta_n(i))) \in \psi_i(\alpha(i))\} \in U\}.$$

Proof: we first show the following Lemma: If 1) holds for constants of types τ_i ($i = 1, \dots, n$), then 2) holds for constants of type $\tau = (\tau_1, \dots, \tau_n)$.

Proof of the Lemma: The inclusion from the right to the left is obvious. We take any $\bar{\alpha} \in \overline{M}^\tau$. Let $(a_1, \dots, a_n) \in \psi(\bar{\alpha})$. Hence there exist $\bar{\alpha}_1, \dots, \bar{\alpha}_n$, $\bar{\alpha}_i \in \overline{M}^{\tau_i}$ ($i = 1, \dots, n$), such that $a_i = \psi(\bar{\alpha}_i)$ and

$$Z = \{i: (\psi_i(\alpha_1(i)), \dots, \psi_i(\alpha_n(i))) \in \psi_i(\alpha(i))\} \in U.$$

We take any $\bar{\beta}_1, \dots, \bar{\beta}_n$ such that $a_i = \psi(\bar{\beta}_i)$ ($i = 1, \dots, n$). Hence $\psi(\bar{\alpha}_i) = \psi(\bar{\beta}_i)$ so that all

$$Z_i = \{i: \psi_i(\alpha_i(i)) = \psi_i(\beta_i(i))\} \in U.$$

We have $Z \cap \bigcap_{i=1}^n Z_i \in U$,

$$Z \cap \bigcap_{i=1}^n Z_i \subset W = \{\iota: (\psi_i(\beta_1(\iota)), \dots, \psi_i(\beta_n(\iota))) \in \psi_i(\alpha(\iota))\}$$

that yields $W \in U$. This completes the proof of the Lemma.

Now we shall prove 1) by induction on τ . The implication from right to the left and the case $\tau = 1$ are obvious. We take any $\tau \neq 1$, $\tau = (\tau_1, \dots, \tau_n)$ and assume 1) holds for constants of types τ_i ($i = 1, \dots, n$). Suppose that

$$\{\iota: \psi_i(\alpha(\iota)) = \psi_i(\beta(\iota))\} \notin U.$$

So that

$$Z = \{\iota: \psi_i(\alpha(\iota)) \neq \psi_i(\beta(\iota))\} \in U.$$

$Z = Z_1 \cup Z_2$ where

$$Z_1 = \{\iota: \psi_i(\alpha(\iota)) \setminus \psi_i(\beta(\iota)) \neq \phi\},$$

$$Z_2 = \{\iota: \psi_i(\beta(\iota)) \setminus \psi_i(\alpha(\iota)) \neq \phi\};$$

so that $Z_1 \in U$ or $Z_2 \in U$. We consider the first case. For each $\iota \in Z_1$ we choose $b' = (b'_1, \dots, b'_n)$ such that $b' \in \psi_i(\alpha(\iota))$ and $b' \notin \psi_i(\beta(\iota))$. Then there exist $\gamma'_i \in \bar{M}_i^{\alpha_i}$ such that $b'_i = \psi_i(\gamma'_i)$ ($i = 1, \dots, n$). We put $\bar{\gamma}_i = [\gamma'_i]$ where $\gamma_i(\iota) = \gamma'_i$ for $\iota \in Z_1$ and an arbitrarily chosen element from $\bar{M}_i^{\alpha_i}$ for $\iota \in I \setminus Z_1$. We take $c = (c_1, \dots, c_n)$ where $c_i = \psi(\bar{\gamma}_i)$ ($i = 1, \dots, n$). Z_1 is a subset of

$$Z_3 = \{\iota: (\psi_i(\gamma_1(\iota)), \dots, \psi_i(\gamma_n(\iota))) \in \psi_i(\alpha(\iota))\}$$

so $Z_3 \in U$. Hence $c \in \psi(\bar{\alpha})$ so that $c \in \psi(\beta)$. Applying the Lemma we have $Z_4 \in U$ where

$$Z_4 = \{\iota: (\psi_i(\gamma_1(\iota)), \dots, \psi_i(\gamma_n(\iota))) \in \psi_i(\beta(\iota))\}$$

Since $Z_4 \subset I \setminus Z_1$ we have $I \setminus Z_1 \in U$ so that $Z_1 \notin U$, which leads to a contradiction. This completes the proof of the first part of the theorem. Now the second part is obviously valid in view of the Lemma.

Theorem 2. The ultraproduct of normal models is also a normal model.

Proof: We take any $\bar{\alpha}, \bar{\beta} \in \bar{M}^\tau$ such that $\psi(\bar{\alpha}) = \psi(\bar{\beta})$. From Theorem 1 we have

$$Z = \{\iota: \psi_i(\alpha(\iota)) = \psi_i(\beta(\iota))\} \in U.$$

The mappings ψ_i are all injections so that

$$Z \subset \{\iota: \alpha(\iota) = \beta(\iota)\} = Z_1$$

Hence $Z_1 \in U$ so $\alpha \sim_U \beta$ and finally $\bar{\alpha} = \bar{\beta}$.

Theorem 3. If U is a principal ultrafilter then for some $\iota_0 \in I$ models M and M_{ι_0} are isomorphic.

Proof: From [2] p. 124 there exist bijections $h: \underline{M} \rightarrow \underline{M}_{i_0}$, $h_\tau: \overline{M}^\tau \rightarrow \overline{M}_{i_0}^\tau$ such that $h([f]) = f(i_0)$ and $h_\tau(\bar{\alpha}) = \alpha(i_0)$ ($\bar{\alpha} = [\alpha] \in \overline{M}^\tau$). We show that $\tau h(\psi(\bar{\alpha})) = \psi_{i_0}(h_\tau(\bar{\alpha}))$ by induction on τ .

$$1) \tau h(\psi(\bar{\alpha})) = h(\psi(\bar{\alpha})) = \psi_{i_0}(\alpha(i_0)) = \psi_{i_0}(h_1(\bar{\alpha})).$$

2) $\tau = (\tau_1, \dots, \tau_n)$. We assume that $\tau_i h(\psi(\bar{\alpha}_i)) = \psi_{i_0}(\alpha_i(i_0))$ for all $\bar{\alpha}_i \in \overline{M}^{\tau_i}$ ($i = 1, \dots, n$).

$$(b_1, \dots, b_n) \in \tau h(\psi(\bar{\alpha}))$$

iff there exist $a_i, \bar{\alpha}_i$ such that $b_i = \tau_i h(a_i)$, $a_i = \psi(\bar{\alpha}_i)$ and

$$\{i: (\psi_i(\alpha_1(i)), \dots, \psi_i(\alpha_n(i))) \in \psi_i(\alpha(i))\} \in U$$

iff

$$(\psi_{i_0}(\alpha_1(i_0)), \dots, \psi_{i_0}(\alpha_n(i_0))) \in \psi_{i_0}(\alpha(i_0))$$

iff

$$(\tau_1 h(\psi(\bar{\alpha}_1)), \dots, \tau_n h(\psi(\bar{\alpha}_n))) \in \psi_{i_0}(\alpha(i_0))$$

iff

$$(b_1, \dots, b_n) \in \psi_{i_0}(\alpha(i_0)).$$

Let $M_0 = (\underline{M}_0, \psi_0)$ be any model, I any set and U any ultrafilter on I . The ultrapower of M_0 is the model $\underline{M}_0^I/U = \underline{M} = (\underline{M}, \psi)$ where $\underline{M} = \underline{M}_0^I/U$, $\overline{M}^\tau = \overline{M}_0^{\tau I}/U$ (see [2], 89) and ψ is defined in the same way as above. d will denote the canonical embedding \underline{M}_0 in \underline{M} ($d(a) = [A(a)]$, where A is a diagonal injection \underline{M}_0 in \underline{M}) and d_τ respectively the embeddings \overline{M}_0^τ in \overline{M}^τ (similarly $d_\tau(\alpha) = [A_\tau(\alpha)]$). We define the model $dM_0 = (d\underline{M}_0, \tilde{\psi})$ as follows: $d\underline{M}_0 = d(\underline{M}_0)$, $d\overline{M}_0^\tau = d_\tau(\overline{M}_0^\tau)$, and for $\bar{\alpha} \in d\overline{M}_0^\tau$ we put

$$\tilde{\psi}(\bar{\alpha}) = \tau d \circ \psi_0 \circ d_\tau^{-1}(\bar{\alpha})$$

It is easy to see that $\tilde{\psi}: d\overline{M}_0^\tau \rightarrow \tau d\underline{M}_0$.

$$\begin{aligned} \tau d\underline{M}_0 &= \{\tilde{\psi}(\bar{\alpha}): \bar{\alpha} \in d\overline{M}_0^\tau\} = \{\tilde{\psi}(d_\tau(\alpha_0)): \alpha_0 \in \overline{M}_0^\tau\} \\ &= \{\tau d(\psi_0(\alpha_0)): \alpha_0 \in \overline{M}_0^\tau\} = \tau d(\tau M_0) \end{aligned}$$

so that $\tau d\underline{M}_0 \subset \tau d\underline{M}_0 \subset \tau \underline{M}$. If M_0 is full then $\tau d\underline{M}_0 = \tau dM_0$. Straight from the construction of dM_0 we find that M_0 and dM_0 are isomorphic models.

Theorem 4. Models M_0 and dM_0 have the following properties:

- 1) $d\underline{M}_0 \subset \underline{M}$;
- 2) $d\overline{M}_0^\tau \subset \overline{M}^\tau$ for each $\tau \in T$;
- 3) $\tilde{\psi}(\bar{\alpha}) = \psi(\bar{\alpha})$ for $\bar{\alpha} \in d\overline{M}_0^1$;
- 4) $\tilde{\psi}(\bar{\alpha}) = \psi(\bar{\alpha}) \cap (d\underline{M}_0)^n$ for $\bar{\alpha} \in d\overline{M}_0^\tau$ where $r(\tau) = 1$, $\tau = (1, \dots, 1)$.

Proof: 1) and 2) are obvious. 3) can be proved simply. To prove 4) we show first that $\tilde{\psi}(\bar{\alpha}) \subset \psi(\bar{\alpha}) \cap (d\underline{M}_0)^n$. $\bar{\alpha} = d_\tau \alpha$ where $\alpha \in \overline{M}_0^\tau$ so that

$$\tilde{\psi}(\bar{\alpha}) = \tau d(\psi_0(\alpha)) = \{(da_1, \dots, da_n): (a_1, \dots, a_n) \in \psi_0(\alpha)\}.$$

Let $(b_1, \dots, b_n) \in \tilde{\psi}(\bar{\alpha})$. We have $b_i = da_i$ ($i = 1, \dots, n$) where $(a_1, \dots, a_n) \in \psi_0(\alpha)$. Hence $a_i = \psi_0(\alpha_i)$ for some α_i . From 3) we have

$$b_i = d(\psi_0(\alpha_i)) = \psi(d_1(\alpha_i)).$$

We take $d_i(\alpha) = [A_i(\alpha)]$ and $d_1(\alpha_i) = [A_1(\alpha_i)]$. Now $(b_1, \dots, b_n) \in \psi(\bar{\alpha})$ because

$$\begin{aligned} \{ \iota: (\psi_0(A_1(\alpha_1)(\iota)), \dots, \psi_0(A_1(\alpha_n)(\iota))) \in \psi_0(A_i(\alpha)(\iota)) \} \\ = \{ \iota: (a_1, \dots, a_n) \in \psi_0(\alpha) \} = I \in U. \end{aligned}$$

Conversely, we suppose that

$$(b_1, \dots, b_n) \in \psi(\bar{\alpha}) \cap (dM_0)^n.$$

Hence $b_i = \psi(\beta_i)$ for some β_i , $b_i = d(a_i)$ for some $a_i \in M_0$ ($i = 1, \dots, n$) and

$$\{ \iota: (\psi_0(\beta_1(\iota)), \dots, \psi_0(\beta_n(\iota))) \in \psi_0(\alpha) \} \in U.$$

We can choose such β_i that $\psi_0(\beta_i(\iota)) \equiv a_i$ hence $(a_1, \dots, a_n) \in \psi_0(\alpha)$ so that

$$(da_1, \dots, da_n) = (b_1, \dots, b_n) \in d\psi_0(\alpha) = \tilde{\psi}(\bar{\alpha}).$$

Remark: If $r(\tau) > 1$ the similar condition to 4) does not hold. For example let $M_0 = (\underline{M}_0, \psi_0)$ be any full model. We take $\alpha \in \bar{M}_0^{(1)}$ such that $\psi_0(\alpha) = \{M_0\}$; $\bar{\alpha} = d_{((1))}(\alpha)$. $b \in \psi(\bar{\alpha})$ iff $b = \psi(\beta)$ for some $\beta \in \bar{M}^{(1)}$ and $\{ \iota: \psi_0(\beta(\iota)) \in \psi_0(\alpha(\iota)) \} \in U$. Since $\{ \iota: \alpha(\iota) = \alpha \} \in U$ we have $\{ \iota: \psi_0(\beta(\iota)) \in \{M_0\} \} \in U$. Hence $\{ \iota: \psi_0(\beta(\iota)) = M_0 \} \in U$ so that $b = \underline{M}_0^I/U$ and finally $\psi(\bar{\alpha}) = \{ \underline{M}_0^I/U \}$. $\psi(\bar{\alpha}) \cap ((1))dM_0 = \emptyset$ but

$$\tilde{\psi}(\bar{\alpha}) = ((1))d(\psi_0(\alpha)) = ((1))d(\{M_0\}) = \{dM_0\}.$$

The question whether

$$\tilde{\psi}(\bar{\alpha}) \supseteq \psi(\bar{\alpha}) \cap (\tau_1 dM_0 \times \dots \times \tau_n dM_0)$$

holds is open.

Theorem 5. We take $M_0 = (N, \psi_0)$ with arbitrary ψ_0 , $I = N$, U — some non-principal ultrafilter on N (N denotes the set of natural numbers). Then the model $M = M_0^I/U$ is not full.

Proof: We shall show that $dN \in \tau M \setminus \tau M$. Suppose that $dN \in \tau M$. Hence there exists $\bar{\alpha} \in \bar{M}^{(1)}$ such that $dN = \psi(\bar{\alpha})$. We consider three cases:

1) $Z_1 = \{ \iota: \psi_0(\alpha(\iota)) \}$ is an infinite subset of N $\in U$. Since U is a non-principal ultrafilter, Z_1 is infinite. Hence there exists the bijection $\mu: N \rightarrow Z_1$. We define the mapping $g: I \rightarrow N$ as follows:

a) on Z_1 by induction:

$$g(\mu(1)) = a_1 \text{ — any element of } \psi_0(\alpha(\mu(1))),$$

$$g(\mu(n+1)) = a_{n+1} \text{ such that } a_{n+1} \in \psi_0(\alpha(\mu(n+1))) \text{ and } a_{n+1} > a_k \text{ for all } k \leq n;$$

b) for $\iota \in I \setminus Z_1$ we put arbitrary numbers.

For each $k \in N$ there exists $\beta_k \in \bar{M}_0^1$ such that $g(k) = \psi_0(\beta_k)$. We take $\bar{\beta} = [\beta]$ where $\beta(k) = \beta_k$. We have $\psi(\bar{\beta}) = [g]$ and $\psi(\bar{\beta}) \in \psi(\bar{\alpha})$, hence $[g] \in dN$. So that there exists

some $n \in N$, such that $[g] = dn$. $W = \{\iota: g(\iota) = n\} \in U$ hence $W \cap Z_1 \in U$. From the construction of g we see that $W \cap Z_1$ contains exactly one element; it is a contradiction to the non-principality of U .

2) $Z_2 = \{\iota: \psi_0(\alpha(\iota))$ is a finite subset of $N\} \in U$. We define the mapping $g: I \rightarrow N$ as follows: $g(\iota) = \max \psi_0(\alpha(\iota))$ for $\iota \in Z_2$ and $g(\iota)$ is an arbitrary number for $\iota \notin Z_2$. In a similar way as before we have $[g] \in dN$ so that there exists $n \in N$ such that $Z = \{\iota: g(\iota) = n\} \in U$. $Z_1 \cap Z \in U$ so

$$\{\iota: \max \psi_0(\alpha(\iota)) = n\} \in U.$$

Now it is easy to verify that $\psi(\bar{\alpha})$ contains at least $n+1$ elements; this is a contradiction, because dN is an infinite set.

3) $Z_3 = \{\iota: \psi_0(\alpha(\iota)) = \emptyset\} \in U$. In this case we obtain $\psi(\bar{\alpha}) = \emptyset$ because if $[g] \in \psi(\bar{\alpha})$ then

$$Z_3 \cap \{\iota: g(\iota) \in \psi_0(\alpha(\iota))\} = \emptyset \in U$$

contradiction.

Theorem 6. (The Łoś theorem for higher-order models). We take $M = \prod_{\iota \in I} M_\iota / U$; F —any formula such that the constants of F are $\{\bar{\alpha}_1, \dots, \bar{\alpha}_k\}$, where $\bar{\alpha}_j \in \bar{M}$, $\bar{\alpha}_j = [\alpha_j]$ ($j = 1, \dots, k$). We put

$$F_i = F(\bar{\alpha}_1/\alpha_1(\iota), \dots, \bar{\alpha}_k/\alpha_k(\iota)).$$

For $\varphi \in \forall \exists \mathcal{L}_M F$, $v \in Fv^c F$ we have $\varphi(v) \in \tau M$. Then there exists $\bar{\alpha} = [\alpha] \in \bar{M}^c$ such that $\varphi(v) = \psi(\bar{\alpha})$. We put $\varphi_i(v) = \psi_i(\alpha(\iota))$. In this situation the following condition holds:

$$M \models_\varphi F \quad \text{iff} \quad \{\iota: M_\iota \models_{\varphi_i} F_i\} \in U.$$

Proof: We shall prove this by induction on F .

1) $F = s(t_1, \dots, t_n)$ i.e. F is an atomic formula.

$$M \models_\varphi s(t_1, \dots, t_n)$$

iff

$$(\varphi_M t_1, \dots, \varphi_M t_n) \in \varphi_M s$$

iff

$$(i) \quad \{\iota: (\psi_i(\alpha_1(\iota)), \dots, \psi_i(\alpha_n(\iota))) \in \psi_i(\alpha(\iota))\} \in U,$$

where $\varphi_M s = \psi(\bar{\alpha})$ and $\varphi_M t_i = \psi(\bar{\alpha}_i)$ ($i = 1, \dots, n$). We put $t'_i = t_i$ if $t_i \in V^{\tau_i}$ and $t'_i = \alpha_i(\iota)$ if $t_i \in \bar{M}^{\tau_i}$ (similarly s'). Then $\varphi_{iM} t'_i = \psi_i(\alpha_i(\iota))$ and $\varphi_{iM} s' = \psi_i(\alpha(\iota))$ ($i = 1, \dots, n$). Thus (i)

iff

$$\{\iota: M_\iota \models_{\varphi_i} s'(t'_1, \dots, t'_n)\} \in U.$$

2) If F is of the form $F = \neg H$ or $F = F_1 \wedge F_2$ this may be proved in the same way as in the elementary case (see [2], 90).

3) We take F of the form $F = \exists v H(v)$ and we assume that the theorem holds for H . $M \models_\varphi \exists v H(v)$ iff for some $b \in \tau M$ ($b = \psi(\bar{\alpha})$ where $\bar{\alpha} \in \bar{M}^c$)

(ii)

$$M \models_{\varphi \mathfrak{S}_{(v,b)}} H(v).$$

