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On Non-Homogeneous Linear Functional Inequality

In this paper we shall deal with the non-homogeneous functional inequality

$$\psi[f(x)] \leqslant g(x)\psi(x)+h(x), \tag{1}$$

where f, g and h are given functions and ψ is an unknown function. The aim of this paper will be to prove a comparison theorem, i.e. to prove that for a given solution ψ of inequality (1), there exists such a solution φ_0 of the functional equation

$$\varphi[f(x)] = g(x)\varphi(x)+h(x), \tag{2}$$

that

$$\psi(x) \geqslant \varphi_0(x). \tag{3}$$

In the sequel, we assume the following hypothesis

(*H*) 1° The function f is defined, continuous and strictly increasing in an interval $[\xi, a)$ and

$$\xi<f(x)<x \quad \text{for} \quad x\in(\xi, a), \; f(\xi) = \xi.$$

2° The function g is defined and continuous in $[\xi, a)$ and

$$g(x)>0 \quad \text{for} \quad x\in[\xi, a). \tag{4}$$

3° There exists such a point $x_0\in(\xi, a)$ that the sequence

$$G_n(x) = \prod_{i=0}^{n-1} g(f^i(x)) \quad \text{for} \quad x\in[\xi, a), n\geqslant 1; \; G_0(x) = 1, \tag{5}$$

where f^i denotes the ith iterate of the function f, converges to zero uniformly in the interval $[f(x_0), x_0]$.

4° The function h is defined and continuous in $[\xi, a)$ and $h(\xi) = 0$.

For the more general case of a non-linear functional inequality a comparison theorem has been proved in [1]. The theorem applies to the inequality (1) in the case where either

$\lim_{n\to\infty} G_n(x) = \infty$ for $x \in [\xi, a)$, or $\lim_{n\to\infty} G_n(x) = G(x)$ for $x \in [\xi, a)$, where G is a continuous function, positive in $[\xi, a)$. The results obtained in [1] cannot be applied, however, to the case where $G_n(x) \to 0$ for an $x \in [\xi, a)$. Such a case has also been considered in [1] and [2] but only for the homogeneous linear inequality.

It is easy to verify by induction the following (cf. [3], p. 53)

Lemma 1. If ψ satisfies the inequality (1) in $[\xi, a)$, then

$$\psi[f^n(x)] \leqslant G_n(x)\psi(x) + G_n(x)\sum_{i=0}^{n-1}\frac{h[f^i(x)]}{G_{i+1}(x)}, \tag{6}$$

for $n = 1, 2, \ldots$ and $x \in [\xi, a)$ and the equality holds iff ψ satisfies equation (2).

We are going to prove the following

Lemma 2. If ψ satisfies the inequality (1) in $[\xi, a)$, then the sequence

$$\psi_n(x) = \frac{\psi[f^n(x)]}{G_n(x)} - \sum_{i=0}^{n-1}\frac{h[f^i(x)]}{G_{i+1}(x)} \tag{7}$$

for $n = 1, 2, \ldots$, $x \in [\xi, a)$ is decreasing.

Proof. Let ψ satisfy the inequality (1). Then

$$\begin{aligned}
&\psi_{n+1}(x) - \psi_n(x) \\
&\quad = \frac{\psi[f^{n+1}(x)]}{G_{n+1}(x)} - \sum_{i=0}^{n}\frac{h[f^i(x)]}{G_{i+1}(x)} - \frac{\psi[f^n(x)]}{G_n(x)} + \sum_{i=0}^{n-1}\frac{h[f^i(x)]}{G_{i+1}(x)} \\
&\quad = \frac{\psi[f^{n+1}(x)] - g[f^n(x)]\psi[f^n(x)] - h[f^n(x)]}{G_{n+1}(x)} \leqslant 0,
\end{aligned}$$

for $n = 1, 2, \ldots$, $x \in [\xi, a)$, because $f^n(x) \in [\xi, a)$, by virtue of hypothesis (H) and, consequently, ψ satisfies the inequality (1) for $f^n(x)$, put in place of x.

We are now able to prove the following

Theorem. Let the hypothesis (H) be fulfilled and let ψ be a continuous solution of inequality (1) in $[\xi, a)$. If there exists such a solution $\bar{\varphi}(x)$ of equation (2) in $[\xi, a)$ that

$$\psi(x) \geqslant \bar{\varphi}(x) \quad \text{for} \quad x \in [\xi, a), \tag{8}$$

and

$$\psi(\xi) = \bar{\varphi}(\xi) = 0,$$

then there exists the limit

$$\varphi_0(x) = \lim_{n\to\infty}\psi_n(x) \quad \text{for} \quad x \in [\xi, a), \tag{9}$$

where ψ_n is defined by the formula (7).

The function φ_0 is upper semi-continuous in (ξ, a), continuous at ξ, and the inequality (3) holds in $[\xi, a)$. Moreover, φ_0 is the greatest solution of equation (2) satisfying (3), i.e. if φ is a solution of (2) in $[\xi, a)$ satisfying the inequality

$$\varphi(x_0) > \varphi_0(x_0) \tag{10}$$

for an $x_0 \in [\xi, a)$, then there exists such a positive integer k that

$$\varphi[f^k(x_0)] > \psi[f^k(x_0)]\,. \tag{11}$$

Proof. First, let us notice that the inequality (8) implies that if $\bar{\varphi}$ satisfies (2), then

$$\psi_n(x) \geqslant \frac{\bar{\varphi}[f^n(x)]}{G_n(x)} - \sum_{i=0}^{n-1} \frac{h[f^i(x)]}{G_{i+1}(x)} = \bar{\varphi}(x)$$

for $n = 1, 2, \ldots, x \in [\xi, a)$, by virtue of (7).

This means that the limit (9) exists, by virtue of lemma 2. It is easy to verify that φ_0 satisfies the equation (2) and that it is an upper semi-continuous function in $[\xi, a)$, as a limit of the decreasing sequence of continuous functions. It follows from the hypothesis (H), (8) and (7) that

$$0 = \varphi_0(\xi) \leqslant \lim_{x\to\xi} \psi_n(x) = 0$$

(see [3]), which shows that the function φ_0 is continuous at ξ. The inequality (3) follows immediately.

Let $x_0 \in [\xi, a)$ and let φ be such a solution of equation (2) in $[\xi, a)$ that the inequality (10) holds. Let us suppose that for every positive integer k we have

$$\varphi[f^k(x_0)] \leqslant \psi[f^k(x_0)]\,. \tag{12}$$

It follows from (12), (7) and (6) of Lemma 1 for φ, that $\varphi(x_0) \leqslant \psi_k(x_0)$ for any $k \geqslant 1$. Letting here $k \to \infty$ we get by (9) a contradiction with (10). This ends the proof of the theorem.

Let us notice that the function φ_0 need not be continuous even if the inequality is homogeneous. To see this consider the following

Example. For the inequality

$$\psi\left(\frac{1}{2}x\right) \leqslant \tfrac{1}{2}\psi(x),\ x \in [0, 1] \tag{13}$$

the assumptions of our theorem are fulfilled. We have $f^n(x) = x2^{-n}$ and $G_n(x) = 2^{-n+1}$, $n = 0, 1, \ldots$ It is easily verified that the function

$$\psi(x) = \begin{cases} x + 2^{n^2+n}(x - 3.2^{-n-2})^n & \text{for} \quad x \in [3.2^{-n-2}, 2^{-n}]\,, \\ x + 2^{n^2+n-1}(3.2^{-n-2} - x)^n & \text{for} \quad x \in (2^{-n-1}, 3.2^{-n-2}), \\ 0 & \text{for} \quad x = 0\,, \\ & n = 0, 1, 2, \ldots \end{cases}$$

is a continuous solution of inequality (13) in the interval [0, 1]. Calculating the limit (9) however, we get

$$\varphi_0(x) = \begin{cases} 2x & \text{for } x = 2^{-n}, n = 1, 2, \dots, \\ x & \text{otherwise (in } [0, 1]) \end{cases}$$

i.e. a function discontinuous in the interval [0, 1].

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