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### Some Theorems on Partial Differential-Functional Inequalities of Parabolic Type

We consider a system of second order differential-functional inequalities of the type

$$(1) \quad u_i^i(t, x) \leq f^i(t, X, U, u_x^i, u_{xx}^i, U(t, \cdot)), \quad (i = 1, \dots, m),$$

where  $X = (x_1, \dots, x_n)$ ,  $U = (u^1, \dots, u^m)$ ,  $u_x^i = (u_{x_1}^i, \dots, u_{x_n}^i)$ ,  $u_{xx}^i$  is the matrix of second order derivatives with respect to  $x$  and for fixed  $t$  we denote by

$$U(t, \cdot) = (u^1(t, \cdot), \dots, u^m(t, \cdot))$$

an element of the space of continuous functions such that  $U(t, \cdot): X \rightarrow U(t, X)$ .

Theorems 2 and 5 are generalizations of results obtained by P. Besala for parabolic differential inequalities [2], [3].

For any vectors  $U = (u^1, \dots, u^m)$ ,  $V = (v^1, \dots, v^m)$  we shall write

$$U \leq V, \text{ if } u^j \leq v^j \text{ (} j = 1, \dots, m \text{).}$$

and

$$U < V, \text{ if } u^j < v^j \text{ (} j = 1, \dots, m \text{).}$$

For a fixed  $i$  we write

$$U^i \leq V^i, \text{ if } u^j \leq v^j \text{ (} j = 1, \dots, m \text{) and } u^i = v^i.$$

A region  $D$  in the space of points  $(t, x_1, \dots, x_n)$  will be called a region of type  $C$  if the following conditions are satisfied:

(a)  $D$  is open, contained in the zone  $t_0 < t < t_0 + T \leq \infty$ , and the intersection of the closure of  $D$  with any closed zone  $t_0 \leq t \leq t_1 < t_0 + T$  is bounded.

(b) The projection  $S_{t_1}$  on the space  $(x_1, \dots, x_n)$  of the intersection of the closure of  $D$  with the plane  $t = t_1$  is, for any  $t_1 \in [t_0, t_0 + T)$ , non-empty.

(c) The point  $(t, X)$  being arbitrarily fixed in the closure of  $D$ , to every sequence  $t_v$  such that  $t_v \in [t_0, t_0 + T)$  and  $t_v \rightarrow t$ , there is a sequence  $X_v$ , so that  $X_v \in S_{t_v}$  and  $X_v \rightarrow X$ .

For a fixed  $t_1 \in [t_0, t_0 + T)$  let  $C_m(S_{t_1})$  stand for the space of continuous functions  $z(X) = (z^1(X), \dots, z^m(X))$  from  $S_{t_1}$  in  $R^m$  with the norms

$$\|z^i\| = \max\{z^i(X) : X \in S_{t_1}\}, \quad \|z\| = \max_{1 \leq i \leq m} \|z^i\|$$

For  $z \in C_m(S_{t_1})$ ,  $\bar{z} \in C_m(S_{t_1})$  we shall write  $z \leq \bar{z}$ , if  $z^j(X) \leq \bar{z}^j(X)$  for arbitrary  $X \in S_{t_1}$  ( $j = 1, \dots, m$ ). For a fixed  $t_1 \in [t_0, t_0 + T)$  we denote by  $U(t, \cdot) = (u^1(t, \cdot), \dots, u^m(t, \cdot))$  an element of the space  $C_m(S_{t_1})$  such that

$$U(t, \cdot): C_m(S_{t_1}) \ni X \rightarrow U(t, X).$$

**Assumptions A.** A region  $D \subset (t, x_1, \dots, x_n)$  of type  $C$  being given let the functions  $\alpha^i(t, X)$  ( $i = 1, \dots, m$ ) be defined and non-negative on its side surface  $\Sigma$  (i.e. that part of the boundary of  $D$  which is contained in the open zone  $t_0 < t < t_0 + T$ ). Denote by  $\Sigma_{\alpha^i}$  the subset of  $\Sigma$  on which  $\alpha^i(t, X) \neq 0$ . For every  $(t, X) \in \Sigma_{\alpha^i}$ , let a direction  $l^i(t, X)$  ( $i = 1, 2, \dots, m$ ) be given, so that  $l^i$  is orthogonal to the  $t$ -axis and some segment starting at  $(t, X)$  of the straight half-line from  $(t, X)$  in the direction  $l^i$  is contained in the closure of  $D$ .

A vector-function  $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$  will be called regular in  $D$  if it is continuous in the closure of  $D$  and possesses continuous derivatives  $\partial/\partial t$ ,  $\partial/\partial x_j$ ,  $\partial^2/\partial x_j \partial x_k$  in  $D$ .

If, in addition, for every  $i$  the derivative  $du^i/dl^i$  exists at each point  $(t, X) \in \Sigma_{\alpha^i}$ , then the vector-function  $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$  is called  $\Sigma_{\alpha}$ -regular in  $D$ .

Let functions  $f^i(t, X, U, Q, R, z)$  ( $i = 1, \dots, m$ ), where  $Q = (q_1, \dots, q_n)$  and  $R = (r_{jk})$  is a  $n \times n$  real symmetric matrix, be defined for  $(t, X) \in D$ ,  $U, Q, R$  arbitrary and  $z \in C_m(S_{t_1})$ .

We shall make use of the following definition of ellipticity given by J. Szarski.

Suppose that  $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$  is defined and possesses first derivatives with respect to  $X$  at a point  $(\bar{t}, \bar{X}) \in D$ .

A function  $f^i(t, X, U, Q, R, z)$  is called elliptic with respect to  $U(t, X)$  at the point  $(\bar{t}, \bar{X}) \in D$  if for any two  $n \times n$  real symmetric matrices  $R = (r_{jk})$ ,  $\bar{R} = (\bar{r}_{jk})$  ( $j, k = 1, \dots, n$ ) such that  $R \leq \bar{R}$  (i.e. such that the quadratic form  $\sum_{j,k} (r_{jk} - \bar{r}_{jk}) \lambda_j \lambda_k$  is non-positive) we have

$$f^i(\bar{t}, \bar{X}, U(\bar{t}, \bar{X}), u_x^i(\bar{t}, \bar{X}), R, U(\bar{t}, \cdot)) \leq f^i(\bar{t}, \bar{X}, U(\bar{t}, \bar{X}), u_x^i(\bar{t}, \bar{X}), \bar{R}, U(\bar{t}, \cdot)).$$

If the above property holds true for every point  $(\bar{t}, \bar{X}) \in D$ , then we say that  $f^i(t, X, U, Q, R, z)$  is elliptic with respect to  $U(t, X)$  in  $D$ .

We shall make use of the following Lemma (see [1], Lemma 47.1).

**Lemma 1.** Suppose we are given a region  $D$  of type  $C$ , a function  $\alpha(t, X)$  and a direction  $l(t, X)$  satisfying (for  $m = 1$ ) Assumptions  $A$  on the side surface  $\Sigma$  of  $D$ , and a function  $\beta(t, X)$  on  $\Sigma_{\alpha}$  such that

$$\beta(t, X) > B \geq 0 \quad \text{for} \quad (t, X) \in \Sigma_{\alpha}.$$

Let the function  $u(t, X)$  be continuous in the closure of  $D$  and possess the derivative  $du/dl$  on  $\Sigma_{\alpha}$ . Suppose that

$$\beta(t, X)u(t, X) - \alpha(t, X) \frac{du}{dl} \leq B\eta(t) (< B\eta(t)) \quad \text{for} \quad (t, X) \in \Sigma_{\alpha},$$

$$u(t, X) \leq \eta(t) (< \eta(t)) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{\alpha},$$

where  $\eta(t) \geq 0$ .

Under these assumptions, if for a point  $(\bar{t}, \bar{X}) \in D$  ( $t_0 < \bar{t} < t_0 + T$ ) we have

$$\max_{x \in \bar{t}} u(\bar{t}, X) = u(\bar{t}, \bar{X}) > \eta(\bar{t}) (\geq \eta(\bar{t})),$$

then  $(\bar{t}, \bar{X})$  is an interior point of  $D$ .

Theorem 1 (see [1], Theorem 63.1.) Assume the functions

$$f^i(t, X, U, Q, R, z) = f^i(t, x_1, \dots, x_n, u^1, \dots, u^m, q_1, \dots, q_n, r_{11}, r_{12}, \dots, r_m, z)$$

$i = 1, 2, \dots, m$ ) to be defined for  $(t, X) \in D$  of type  $C$ , for arbitrary  $U, Q, R$ , for  $z \in C_m(S_i)$  and for every fixed index  $i$  the following implication holds true:  $U \leq \bar{U}, z \leq \bar{z}, (t, X) \in D, z, \bar{z} \in C_m(S_i) \Rightarrow f^i(t, X, U, Q, R, z) \leq f^i(t, X, \bar{U}, Q, R, \bar{z})$ . Let the functions  $\alpha^i(t, X)$  and the directions  $l^i(t, X)$  ( $i = 1, 2, \dots, m$ ) satisfy Assumptions  $A$  on the side surface of  $D$ . Suppose  $\beta^i(t, X)$  ( $i = 1, 2, \dots, m$ ) are defined and positive on  $\Sigma_{\alpha^i}$ . Let  $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$  and  $V(t, X) = (v^1(t, X), \dots, v^m(t, X))$  be  $\Sigma_{\alpha}$ -regular in  $D$  and suppose that every function  $f^i$  is elliptic with respect to the vector-function  $U(t, X)$ . Put

$$G^i = \{(t, X) \in D: U(t, X) \leq V(t, X)\} \quad (i = 1, 2, \dots, m)$$

and suppose that, for every fixed  $j$ , we have

$$(1) \quad u_t^j(t^*, X^*) < f^j(t^*, X^*, U(t^*, X^*), u_x^j(t^*, X^*), u_{xx}^j(t^*, X^*), U(t^*, \cdot)),$$

$$(2) \quad v_t^j(t^*, X^*) \geq f^j(t^*, X^*, V(t^*, X^*), v_x^j(t^*, X^*), v_{xx}^j(t^*, X^*), V(t^*, \cdot)),$$

whenever  $(t^*, X^*) \in G^j$ . Suppose finally that the initial inequalities

$$(3) \quad U(t_0, X) < V(t_0, X) \quad \text{for } S_{t_0}$$

and boundary inequalities

$$\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - \alpha^i(t, X) \frac{d[u^i - v^i]}{dt} < 0 \quad \text{for } (t, X) \in \Sigma_{\alpha^i},$$

$$(4) \quad u^i(t, X) - v^i(t, X) < 0 \quad \text{for } (t, X) \in \Sigma - \Sigma_{\alpha^i} \quad (i = 1, 2, \dots, m)$$

hold true.

Under the above assumptions we have

$$(5) \quad U(t, X) < V(t, X)$$

in  $D$ .

Proof. Since the set of points  $(t_0, X)$ , such that  $X \in S_{t_0}$ , is compact, there is, by (3) and by the continuity, a  $\bar{t}$  ( $t_0 < \bar{t} < t_0 + T$ ), so that (5) holds true in the intersection of  $\bar{D}$  with the zone  $t_0 \leq t < \bar{t}$ . Denote by  $t^*$  the least upper bound of such  $\bar{t}$ . We have to prove that  $t^* = t_0 + T$ . Suppose the contrary, i.e.  $t^* < t_0 + T$ . Then, in virtue of the definitions of  $t^*$  and of the region of type  $C$ , we have in  $\bar{D}$

$$(6) \quad U(t, X) \leq V(t, X) \quad \text{for } t_0 \leq t \leq t^*$$

and for some index  $j$  and some  $X^* \in S_{t^*}$

$$(7) \quad u^j(t^*, X^*) = v^j(t^*, X^*).$$

From (6) and (7) it follows that

$$\max_{x \in S_{t^*}} [u^j(t^*, X) - v^j(t^*, X)] = u^j(t^*, X^*) - v^j(t^*, X^*) = 0$$

and hence, by (4) and by Lemma 1, we conclude that  $(t^*, X^*)$  is an interior point of  $D$ . Moreover, by (6) and (7), we have  $(t^*, X^*) \in G^j$ , and consequently inequalities (1) and (2) hold true. The difference  $u^j(t^*, X) - v^j(t^*, X)$  is of class  $C^2$  and attains its maximum at the interior point  $X^*$ . Therefore, we have

$$(8) \quad u_x^j(t^*, X^*) = v_x^j(t^*, X^*)$$

and the quadratic form in  $\lambda_1, \dots, \lambda_n$

$$(9) \quad \sum_{l,k=1}^n [u_{x_l x_k}^j(t^*, X^*) - v_{x_l x_k}^j(t^*, X^*)] \lambda_l \lambda_k$$

is non-positive. Now, from (1), (2) and (8) it results that

$$u_t^j(t^*, X^*) - v_t^j(t^*, X^*) < f^j(t^*, X^*, U(t^*, X^*), u_x^j(t^*, X^*), u_{xx}^j(t^*, X^*), U(t^*, \cdot)) - \\ - f^j(t^*, X^*, V(t^*, X^*), u_x^j(t^*, X^*), v_{xx}^j(t^*, X^*), V(t^*, \cdot)).$$

By (6), (7) and by the assumption on the monotonicity of function  $f^j$  with respect to  $u^1, \dots, u^{j-1}, \dots, u^m, z$  we get from the last inequality

$$u_t^j(t^*, X^*) - v_t^j(t^*, X^*) < f^j(t^*, X^*, U(t^*, X^*), u_x^j(t^*, X^*), u_{xx}^j(t^*, X^*), U(t^*, \cdot)) - \\ - f^j(t^*, X^*, U(t^*, X^*), u_x^j(t^*, X^*), v_{xx}^j(t^*, X^*), U(t^*, \cdot)).$$

Owing to the ellipticity of  $f^j$  with regard to  $U(t, X)$  and by (9), the right side of the last inequality is non-positive and consequently we have

$$(10) \quad u_t^j(t^*, X^*) - v_t^j(t^*, X^*) < 0.$$

On the other hand, the function

$$u^j(t, X^*) - v^j(t, X^*)$$

of one variable  $t$  attains, by (6) and (7), its maximum at the right-hand side extremity  $t^*$  of the interval  $[t_0, t^*]$ . Hence it follows that

$$u_t^j(t^*, X^*) - v_t^j(t^*, X^*) \geq 0,$$

which contradicts (10). This completes the proof.

**Remark.** Theorem 1 is true if, instead of the ellipticity with regard to  $U(t, X)$ , we assume the ellipticity with respect to  $V(t, X)$ .

We shall make use of the following definition of Condition  $C$  introduced in [2] by P. Besala.

Condition C. The index  $i$  being fixed the function  $f^i(t, X, U, Q, R, z)$  will be said to satisfy Condition C with respect to  $u^i$  if  $u^i \leq \tilde{u}^i$  implies

$$f^i(t, X, u^1, \dots, u^{i-1}, u^i, u^{i+1}, \dots, u^m, Q, R, z) - f^i(t, X, u^1, \dots, u^{i-1}, \tilde{u}^i, u^{i+1}, \dots, u^m, Q, R, z) \leq \sigma(t, u^i - \tilde{u}^i),$$

where the function  $\sigma(t, z)$  has the following properties:

- (a)  $\sigma(t, z)$  is continuous and non-negative in the half-strip  $t \in [0, T)$ ,  $z \leq 0$  and  $\sigma(t, 0) \equiv 0$   
 (b) the left-hand minimum solution of the equation

$$\frac{dz}{dt} = \sigma(t, z)$$

satisfying the condition  $\lim_{t \rightarrow T^-} z(t) = 0$  is  $z(t) \equiv 0$ .

We have the following Lemma proved in [2].

Lemma 2. Let  $z_0$  be any fixed positive number. If the function  $\sigma(t, z)$  has properties (a) and (b), then for every  $\varepsilon > 0$  there is  $\delta_0(\varepsilon) > 0$  such that for any  $0 < \delta < \delta_0$  the right-hand minimum solution  $\omega(t)$  of the equation

$$(11) \quad \frac{d\omega}{dt} = -\sigma(t, -\omega) - \delta$$

through  $(0, z_0)$  exists and is positive in  $[0, T - \varepsilon)$ .

Theorem 2 (see [3], Theorem 1).

Assume the functions

$$f^i(t, X, U, Q, R, z) = f^i(t, x_1, \dots, x_n, u^1, \dots, u^m, q_1, \dots, q_n, r_{11}, \dots, r_{nn}, z)$$

( $i = 1, 2, \dots, m$ ) to be defined for  $(t, X) \in D$  of type C (with  $t_0 = 0$ ), for arbitrary  $U, Q, R$  for  $z \in C_m(S_t)$ . For every fixed index  $i$  let the following implication hold true:

$$U \leq \tilde{U}, z \leq \tilde{z}, (t, X) \in D, z, \tilde{z} \in C_m(S_t) \Rightarrow f^i(t, X, U, Q, R, z) \leq f^i(t, X, \tilde{U}, Q, R, \tilde{z}).$$

Let the functions  $\alpha^i(t, X)$  and the directions  $l^i(t, X)$  ( $i = 1, 2, \dots, m$ ) satisfy Assumptions A on the side surface of  $D$ . Suppose  $\beta^i(t, X)$  ( $i = 1, 2, \dots, m$ ) are defined, positive and bounded on  $\Sigma_{\alpha^i}$ . Let  $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$  and  $V(t, X) = (v^1(t, X), \dots, v^m(t, X))$  be  $\Sigma_{\alpha^i}$ -regular in  $D$  and satisfy the initial inequalities

$$(12) \quad U(0, X) < V(0, X) \quad \text{for } X \in S_0$$

and the boundary inequalities

$$u^i(t, X) < v^i(t, X) \quad \text{for } (t, X) \in \Sigma - \Sigma_{\alpha^i}$$

$$(13) \quad \beta^i(t, X) [u^i(t, X) - v^i(t, X)] - \alpha^i(t, X) \frac{d[u^i - v^i]}{dt} \leq -\eta$$

for  $(t, X) \in \Sigma_{\alpha^i}$  ( $i = 1, 2, \dots, m$ )  $\eta$  being a positive constant. Define

$$(14) \quad G = \{(t, X) \in D: U(t, X) \leq V(t, X)\}$$

and let the inequalities

$$(15) \quad u_i^i(t^*, X^*) \leq f^i(t^*, X^*, U(t^*, X^*), u_x^i(t^*, X^*), u_{xx}^i(t^*, X^*), U(t^*, \cdot))$$

$$(i = 1, 2, \dots, m)$$

$$(16) \quad v_i^i(t^*, X^*) \geq f^i(t^*, X^*, V(t^*, X^*), v_x^i(t^*, X^*), v_{xx}^i(t^*, X^*), V(t^*, \cdot))$$

be satisfied whenever  $(t^*, X^*) \in G$ . We assume that every function  $f^i(t, X, U, Q, R, z)$  ( $i = 1, 2, \dots, m$ ) is elliptic with respect to  $V(t, X)$  and satisfies Condition C with respect to  $u^i$ . Under these assumptions the inequality

$$U(t, X) < V(t, X)$$

holds true for  $(t, X) \in D$ .

Proof.  $\varepsilon > 0$  being chosen arbitrarily let  $\Sigma_{\alpha^i}^\varepsilon, (\Sigma - \Sigma_{\alpha^i})^\varepsilon$  be the parts of  $\Sigma_{\alpha^i}, \Sigma - \Sigma_{\alpha^i}$  respectively, which are contained in the zone  $0 < t < T - \varepsilon$ .

Put

$$z_1 = \min \left\{ \inf_{j,k} \inf_{S_0 \cup (\Sigma - \Sigma_{\alpha^i})^\varepsilon} [v^j(t, X) - u^k(t, X)], \inf_{\Sigma_{\alpha^k}^\varepsilon} \eta [\beta^k(t, X)]^{-1} \right\}.$$

It follows from our assumptions that  $z_1 > 0$ . In Lemma 2 we choose  $0 < z_0 < z_1$  and  $\delta$  so that  $\omega(t) > 0$  in  $[0, T - \varepsilon]$ . Observe that  $\omega(t) \leq z_0 < z_1 \leq \eta [\beta^i(t, X)]^{-1}, (t, X) \in \Sigma_{\alpha^i}^\varepsilon$ . Hence, denoting  $\Omega(t) = (\underbrace{\omega(t), \dots, \omega(t)}_m), \bar{u}^i(t, X) = u^i(t, X) + \omega(t), \bar{U}(t, X) = U(t, X) + \Omega(t)$ , we

get from (12), (13)

$$(17) \quad \bar{U}(0, X) < V(0, X) \quad \text{for } X \in S_0,$$

$$(18) \quad \bar{u}^i(t, X) < v^i(t, X) \quad \text{for } (t, X) \in \Sigma - \Sigma_{\alpha^i}^\varepsilon$$

and

$$(19) \quad \beta^i(t, X) [\bar{u}^i(t, X) - v^i(t, X)] - \alpha^i(t, X) \frac{d[\bar{u}^i - v^i]}{dt} < 0$$

for  $(t, X) \in \Sigma_{\alpha^i}^\varepsilon$  ( $i = 1, 2, \dots, m$ ).

Let

$$\bar{G}^i = \{(t, X) \in D: \bar{U}(t, X) \leq V(t, X)\}.$$

We have  $\bar{G}^i \subset G$  ( $i = 1, 2, \dots, m$ ) and consequently inequalities (15), (16) hold true for  $(t^*, X^*) \in \bar{G}^i$ . Now adding (15) and (11) and applying successively condition C and a suitable assumption for the function  $f^i$  we obtain

$$\begin{aligned} \bar{u}_i^i &\leq f^i(t, X, U, u_x^i, u_{xx}^i, U(t, \cdot)) - \sigma(t, -\omega) - \delta \\ &\leq f^i(t, X, u^1, \dots, u^{i-1}, \bar{u}^i, u^{i+1}, \dots, u^m, u_x^i, u_{xx}^i, U(t, \cdot)) - \\ &\quad - \delta \leq f^i(t, X, \bar{U}, \bar{u}_x^i, \bar{u}_{xx}^i, \bar{U}(t, \cdot)) - \delta \end{aligned}$$

that is, since  $\delta > 0$

$$(20) \quad \tilde{u}_i^i < f^i(t, X, \tilde{U}, \tilde{u}_x^i, \tilde{u}_{xx}^i, \tilde{U}(t, \cdot)) \quad \text{for } (t, X) \in \tilde{G}^i.$$

Taking into account (17), (18), (19), (20) and (16) we see that all the assumptions of Theorem 1 (see remark) are satisfied for  $(t, X) \in D$ ,  $0 \leq t < T - \varepsilon$  with  $U(t, x)$  replaced by  $\tilde{U}(t, x)$ ; hence, we have  $\tilde{U}(t, x) < V(t, x)$  for  $(t, x) \in D$ . Since  $\omega(t) > 0$  and  $\varepsilon$  is arbitrary, this completes the proof.

A system of differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \dots, y_m) \quad (i = 1, \dots, m)$$

will be called a comparison system of type I if its right-hand sides are continuous and non-negative and for every fixed index  $i$  the following implication holds true:

$$Y = (y_1, \dots, y_m) \leq \tilde{Y} = (\tilde{y}_1, \dots, \tilde{y}_m) \Rightarrow \sigma_i(t, Y) \leq \sigma_i(t, \tilde{Y})$$

in the closed region:  $t \geq 0$ ,  $y_i \geq 0$  ( $i = 1, 2, \dots, m$ ).

Theorem 3. Let the functions

$$f^i(t, X, U, Q, R, z) = f^i(t, x_1, \dots, x_n, u^1, \dots, u^m, q_1, \dots, q_n, r_{11}, \dots, r_{nn}, z) \quad (i = 1, 2, \dots, m)$$

be defined for  $(t, X) \in D$  of type C, for arbitrary  $U, Q, R$ , for  $z \in C_m(S_t)$  and for every fixed  $i$  let the function  $f^i(t, X, U, Q, R, z)$  be increasing with respect to  $u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^m, z$ . Suppose further that

$$(21) \quad f^i(t, X, U, Q, R, z) - f^i(t, X, \tilde{U}, Q, R, \tilde{z}) \leq \sigma_i(t - t_0, \max(u^1 - \tilde{u}^1, \|z^1 - \tilde{z}^1\|), \dots, \max(u^m - \tilde{u}^m, \|z^m - \tilde{z}^m\|)) \quad (i = 1, 2, \dots, m)$$

whenever  $U \geq \tilde{U}$ ,  $z \geq \tilde{z}$  where  $\sigma_i(t, V)$  are the right-hand sides of a comparison system of type I. As to the comparison system we assume that

$$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \dots, m)$$

and that for its right-hand maximum solution (see [1], § 5) through the origin  $\Omega(t; 0)$  we have

$$\Omega(t; 0) \equiv 0.$$

Let the functions  $\alpha^i(t, X)$  and the directions  $\beta^i(t, X)$  ( $i = 1, 2, \dots, m$ ) satisfy Assumptions A on the side surface  $\Sigma$  of  $D$ . Suppose  $\beta^i(t, X)$  is positive on  $\Sigma_{\alpha^i}$  ( $i = 1, \dots, m$ ). Let  $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$  and  $V(t, X) = (v^1(t, X), \dots, v^m(t, X))$  be  $\Sigma_{\alpha^i}$ -regular in  $D$  and suppose that every function  $f^i(t, X, U, Q, R, z)$  is elliptic with regard to  $U(t, X)$ . Assume that the initial inequality

$$(22) \quad U(t_0, X) \leq V(t_0, X) \quad \text{for } X \in S_{t_0}$$

and boundary inequalities

$$\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - \alpha^i(t, X) \frac{d[u^i - v^i]}{dt} \leq 0 \quad \text{for } (t, X) \in \Sigma_{\alpha^i}$$

$$(23) \quad u^i(t, X) - v^i(t, X) \leq 0 \quad \text{for } (t, X) \in \Sigma - \Sigma_{\alpha^i} \quad (i = 1, \dots, m)$$

are satisfied. Write

$$E^i = \{(t, X) \in D: u^i(t, X) > v^i(t, X)\} \quad (i = 1, 2, \dots, m)$$

and suppose that for every fixed  $j$

$$(24) \quad u_t^j(t^*, X^*) \leq f^j(t^*, X^*, U(t^*, X^*), u_x^j(t^*, X^*), u_{xx}^j(t^*, X^*), U(t^*, \cdot))$$

$$(25) \quad v_t^j(t^*, X^*) \geq f^j(t^*, X^*, V(t^*, X^*), v_x^j(t^*, X^*), v_{xx}^j(t^*, X^*), V(t^*, \cdot)),$$

whenever  $(t^*, X^*) \in E^j$ .

This being assumed, we have in  $D$

$$(26) \quad U(t, X) \leq V(t, X).$$

**Proof.** Since the assumptions of our theorem are invariant under the mapping  $\tau = t - t_0$ , we may assume, without loss of generality, that  $t_0 = 0$ . Put, for  $0 \leq t < T$ ,  $M^i(t) = \max_{X \in S_t} [u^i(t, X) - v^i(t, X)]$ ,  $\tilde{M}^i(t) = \max(0, M^i(t))$  ( $i = 1, 2, \dots, m$ )

$$\tilde{M}(t) = (\tilde{M}^1(t), \dots, \tilde{M}^m(t)).$$

It is clear that the assertion of our theorem is equivalent to the inequality

$$(27) \quad \tilde{M}(t) \leq 0 \quad \text{on } [0, T].$$

We are going to prove relation (27) by means of the first comparison theorem (see [1], §14). By (22), we have  $\tilde{M}(0) \leq 0$  and, by a suitable theorem (see [1], §33) the functions  $\tilde{M}^i(t)$  are continuous on  $[0, T]$ . Therefore, writing

$$\tilde{E}^i = \{t \in (0, T): \tilde{M}^i(t) > 0\} \quad (i = 1, 2, \dots, m),$$

inequality (27) will be proved by the first comparison theorem, if we show that

$$D_- \tilde{M}^i(t) \leq \sigma_i(t, \tilde{M}(t)) \quad \text{for } t \in \tilde{E}^i.$$

Now, fix an index  $j$  and let  $t^* \in E^j$ . By the argument used in [1], Theorem 64.1, there is a point  $X^* \in S_{t^*}$  such that

$$(28) \quad \tilde{M}^j(t^*) = u^j(t^*, X^*) - v^j(t^*, X^*) > 0$$

and

$$(29) \quad D_- \tilde{M}^j(t^*) \leq u_t^j(t^*, X^*) - v_t^j(t^*, X^*).$$

Inequality (28) implies that  $(t^*, X^*) \in E^j$  and consequently by (24), (25) and by the fact that the function  $u^j(t^*, X) - v^j(t^*, X)$  attains its maximum at the interior point  $(t^*, X^*)$  (see [1], Theorem 64.1), we get

