

Geodesics of order 2

by J. GANCARZEWICZ

1. Introduction. Let M be a manifold. $L^r M$ denotes the (principal) fibre bundle of frames of order r . We consider a manifold F and an action on the left of the group L_n^r ($n = \dim M$) on F . A mapping $A: L^r M \rightarrow F$ such that

$$A \circ R_\xi = \lambda_{\xi^{-1}} \circ A, \quad \text{for } \xi \in L_n^r;$$

where $\lambda_\xi: F \rightarrow F$ is a left translation, is called a geometrical object of the type (F, λ_ξ) , or shortly (F, λ_ξ) -object, on $L^r M$.

Let $T^r(n)$ denote the set of all r -jets (at 0) of curves $\gamma: (-\varepsilon, +\varepsilon) \rightarrow \mathbb{R}^n$ such that $\gamma(0) = 0$. $(T^r(n), \lambda_\xi)$ -objects on $L^r M$, where $\lambda_\xi: T^r(n) \rightarrow T^r(n)$ is a natural left translation on $T^r(n)$, are called fields of tangent elements of order r on M . It is easy to see that fields of tangent elements of order 1 are vector fields.

Each curve $\gamma: (-\varepsilon, +\varepsilon) \rightarrow M$ defines along γ the fields $J^r(\gamma)$ of tangent elements of order r . If \bar{F} is a connection of order r on M then a curve γ is called an r -geodesics if

$$D_\gamma^r(J^r(\gamma)) = 0,$$

where D_γ^r denotes an absolute derivative along γ with respect to \bar{F} . In the case $r = 1$, $J^1(\gamma) = \dot{\gamma}$ is a vector field of velocities and our definition coincides with the classical one.

In this note we discuss the problem of the existence of connections of order 2 such that there are 2-geodesics with given initial conditions.

2. An absolute derivative. Let $P(M, G)$ be a principal fibre bundle and F be a manifold on which the group G acts on the left. We denote by $\lambda_\xi: F \rightarrow F$ a left translation on F in this action. An (F, λ_ξ) -object on $P(M, G)$ is a mapping $A: P \rightarrow F$ such that

$$A \circ R_\xi = \lambda_{\xi^{-1}} \circ A \quad \text{for all } \xi \in G.$$

It is not difficult to verify (see Lemma 1 in [1]) that there is 1-1 correspondence between the set of all (F, λ_ξ) -objects on $P(M, G)$ and the set of all sections of the fibre bundle $E = E(M, F, G; P)$ over M , with standard fibre F and

structure group G , which is associated with $P(M, G)$. We recall only (see pp. 54–55 in [2]) that E is a factor set of $P \times F$ by the relation " \sim ", where

$$(2.1) \quad (p, f) \sim (p', f') \Leftrightarrow \exists \xi \in G: p' = p \cdot \xi, f' = \lambda_{\xi^{-1}}(f).$$

If $A: P \rightarrow F$ is an (F, λ_ξ) -object and $\dot{A}: M \rightarrow E$ is a section of the associated fibre bundle then the correspondence between A and \dot{A} is given by the formula

$$(2.2) \quad \dot{A}(\pi(p)) = \langle p, A(p) \rangle,$$

where $\pi: P \rightarrow M$ is the projection in $P(M, G)$ and $\langle p, f \rangle$ denotes the class of (p, f) in E .

Let Γ be a connection in $P(M, G)$ and A an (F, λ_ξ) -object on $P(M, G)$. A covariant derivative $\nabla_v^\Gamma A$ of A with respect to Γ and in the direction of a vector field v (v is a vector field on M) is defined as the composition (see [1])

$$(2.3) \quad \nabla_v^\Gamma A = dA \circ H_v^\Gamma: P \xrightarrow{H_v^\Gamma} TP \xrightarrow{aA} TF,$$

where H_v^Γ is a horizontal lift of v with respect to Γ , that is, $H_v^\Gamma: P \rightarrow TP$ is a vector field on P such that $d\pi \circ H_v^\Gamma = v \circ \pi$ and $H_v^\Gamma(p) \in \Gamma_p$.

It is easy to see (cf. [1]) that $\nabla_v^\Gamma A$ is an $(TF, d\lambda_\xi)$ -object on $P(M, G)$, e.i.,

$$(\nabla_v^\Gamma A) \circ R_\xi = d\lambda_{\xi^{-1}} \circ (\nabla_v^\Gamma A).$$

It is also clear that

$$(2.4) \quad p \circ \nabla_v^\Gamma A = A,$$

where $p: TF \rightarrow F$ is a projection in the bundle TF .

If A is an (F, λ_ξ) -object on $P(M, G)$ then a $(TF, d\lambda_\xi)$ -object B on $P(M, G)$ is called a concomitant of A (of the first order) if $p \circ B = A$. The formula (2.4) means that $\nabla_v^\Gamma A$ is a concomitant of A . We denote by $\Phi(P, A)$ the set of all concomitants of A . $\Phi(P, A)$ is a $C^\infty(M)$ -module. For $B_1, B_2 \in \Phi(P, A)$ and $f_1, f_2 \in C^\infty(M)$ we set

$$(f_1 B_1 + f_2 B_2)(p) = f_1(\pi(p))B_1(p) + f_2(\pi(p))B_2(p).$$

This sum is well defined because $B_1(p)$ and $B_2(p)$ are two vectors which belong to the same fibre of TF , namely, $B_1(p)$ and $B_2(p)$ belong to $T_{A(p)}F$. It is easy to see that the mapping

$$(2.5) \quad \mathfrak{X}(M) \ni v \rightarrow \nabla_v^\Gamma A \in \Phi(P, A)$$

is $C^\infty(M)$ -linear. $\mathfrak{X}(M)$ denotes a $C^\infty(M)$ -module of vector fields on M .)

Let $x_0 \in M$ and $v_0 \in T_{x_0}M$. For an (F, λ_ξ) -object on $P(M, G)$ we define

$$(2.6) \quad \nabla_{v_0}^\Gamma A = (\nabla_{v_0}^\Gamma A)(x_0),$$

