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Zero points of the eigenfunctions of a composition of operators of the Sturm-Liouville type

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1. Introduction. Consider operator $L = L_1 \dots L_k$ as a composition of k operators of Sturm-Liouville type. The main theme of this paper is a quantitative analysis of zero points of the eigenfunction of L. By reasoning of a general character we have proved in § 3 that the zero points of any eigenfunction are of the first order which implies that the eigenvalues are also of the first order. On the basis of this fact we have proved that for any two eigenvalues $\lambda, \mu (|\lambda| < |\mu|)$ and their corresponding eigenfunctions u, v the number of zero points of u is smaller than that of v. Moreover if $K = K_{k-1} K_{k-2} \dots K_1$ where $K_j = L_j^{-1}$ is a symmetric and positive operator then the space $L^2([a, b])$ is unitary with the scalar product $[u, v]_K = (Ku, v)_{L^2}$. Denote by L_k^2 its complement to the complete space. We easily conclude that any two eigenfunctions of L are either linearly dependent or orthogonal in L_k^2 and that $K_k K$ after completing to L_K^2 is self-adjoint and positive in L_K^2. It follows (see e.g. [7]) that there exists an infinite decreasing sequence $\{\lambda_n\}$ of eigenvalues and a corresponding sequence of eigenfunctions $\{u_n\}$ of $K_k K$, complete in L_k^2.

It is not known, however, whether the eigenfunctions of $K_k K$ are also eigenfunctions of L as in the case of a Sturm-Liouville type operator of the second order. Assuming the above sentence holds we have proved that the n-th eigenfunction of L has exactly $n-1$ zeros in (a, b). This result was known for $k = 1$.

The problem is very simple when $L_1, \dots, L_k$ commute. In this case any eigenfunction of the operator $L = L_1 \dots L_k$ is an eigenfunction of L_j, $j = 1, \dots, k$, and vice versa. Hence for commuting operators the problem is reduced to investigation of the eigenfunctions of Sturm-Liouville type operatores of the second order.

2. Operators of the Sturm-Liouville type. Let $\mathcal{M}$ be the set of all functions of class C^2 in $[a, b]$ which satisfy the following boundary conditions

$$\alpha_1 u(a) - \alpha_2 u'(a) = 0$$

$$\beta_1 u(b) + \beta_2 u'(b) = 0 \tag{2.1}$$

where $\alpha_1 > 0$; $\beta_1 > 0$, $\alpha_2 \geqslant 0$, $\beta_2 \geqslant 0$.

Operator $L: \mathcal{M} \to L^2([a,b])$ is said to be of Sturm-Liouville type if

$$Lu(t) = -\big(p(t)u'(t)\big)' + q(t)u(t)\,, \tag{2.2}$$

where $p \in C^1([a,b])$, $q \in C^0([a,b])$.

Definition. The value λ is said to be an eigenvalue of L if the equation $Lu = \lambda u$ has a non zero solution u in $\mathcal{M}$; u is then called an eigenfunction of L corresponding to λ. Throughout the present paper we assume that $p(t) > 0$, $q(t) \geqslant 0$ for $t \in [a,b]$.

The following theorem holds true

THEOREM 2.1. ([5], p. 398) *There exists an unlimited sequence of real numbers* $\lambda_1 < \lambda_2 < \ldots$ *such that*

1. λ *is an eigenvalue of* $L \Leftrightarrow$ *there exists* n *such that* $\lambda = \lambda_n$.
2. *Eigenfunctions corresponding to the same eigenvalue are linearly dependent.*
3. *If* u_n *is an eigenfunction corresponding to the eigenvalue* λ_n *then* u_n *has exactly* $n-1$ *zeros in* (a,b).
4. *If* $n \neq m$ *then* u_n *and* u_m *are orthogonal in* $L^2([a,b])$ *i.e.* $\int\limits_a^b u_n(t)u_m(t)\,dt = 0$.
5. *If we choose* u_n *such that* $\int\limits_a^b u_n^2(t)\,dt = 1$ *then the sequence* $u_1, u_2, \ldots$ *is orthonormal and complete in* $L^2([a,b])$.
6. ([9]) *There exists an operator* K *inverse to* L *and* K *in an integral operator of the Fredholm type with a positive, symmetric and continuous kernel* $G(t,s)$ $(a \leqslant t, s \leqslant b)$. *The domain of* K *is* $C^0([a,b])$.

Operators L *and* $K = L^{-1}$ *are symmetric i.e.*

$$(\varphi, L\psi)_{L^2} = (L\varphi, \psi)_{L^2} \quad \textit{for } \varphi,\psi \in D_L$$

and

$$(\varphi, K\psi)_{L^2} = (K\varphi, \psi)_{L^2} \quad \textit{for } \varphi,\psi \in D_K\,.$$

In what follows we shall need the lemmas

LEMMA 2.1. ([3], p. 44) *If* $f \in C^2([a,b])$ *satisfies the boundary condition* (2.1) *for* $\alpha_2 = 0$ *or* $\beta_2 = 0$ *and* $f(t) \neq 0$ *for* $t \in (a,b)$ *then there exists* $\eta \in (a,b)$ *such that*

$$f(\eta)Lf(\eta) > 0\,.$$

We shall use the following definition of the order of the zero point of f.

Definition. We say that $t_0 \in (a,b)$ is the zero point of f of the order n if for $t \to t_0$

$$\frac{f(t)}{(t-t_0)^j} \to 0 \text{ for } j = 0, 1, \ldots, n-1 \text{ and } \frac{f(t)}{(t-t_0)^n} \not\to 0$$

If f is of class C^n in the neighbourhood of t_0 the above definition is equivalent to the following condition

$$f(t_0) = f'(t_0) = \ldots = f^{(n-1)}(t_0) = 0 \text{ and } f^{(n)}(t_0) \neq 0\,.$$

Applying Taylor's formula we can prove the following

LEMMA 2.2. *If f is of class C^N $(2 \leqslant N < \infty)$ in the neighbourhood of $t_0 \in [a, b]$ and if t_0 is the zero point of f of the order N then*

$$\lim_{t \to t_0} \frac{(t-t_0)^2 Lf(t)}{f(t)} = -N(N-1)p(t_0).$$

Suppose f to be continuous in $[a, b]$. Consider the set

$$A = \{t \in (a, b) \colon f(t) = 0\}$$

A consists of separate intervals closed in (a, b). Let us divide the set M of all intervals into two separate parts M_1 and M_2. M_1 contains one-points intervals and such intervals which are not closed in (a, b), $M_2 = M - M_1$.

Put

$$Z(f) = \begin{cases} l + 2k & \text{if } M_1 \text{ and } M_2 \text{ are finite} \\ \infty & \text{if } M_1 \text{ or } M_2 \text{ is infinite} \end{cases}$$

where l stands for the power of M_1 and k for the power of M_2. In the case of the continuous function f with a finite number of isolated zeros in (a, b), $Z(f)$ denotes the number of zeros of f in (a, b).

Now we prove the following

LEMMA 2.3. *Let $f \in \mathcal{M}$, $Z(f) = p$ $(p < \infty)$; the zero points of f are isolated and of finite order. Suppose for every zero $t_j \in [a, b]$ of order l_j there exists a neighbourhood $V \subset [a, b]$ of t_j in which f is of class C^{2l_j}. Then*

$$Z(Lf) \geqslant p + k$$

where k stands for the number of zero points of f in $[a, b]$ of an order greater than one.

Proof. Denote by $t_1, \ldots, t_p$ $(a < t_1 < \ldots t_p < b)$ the zeros of f in (a, b) and put $t_0 = a$, $t_{p+1} = b$. In every interval

$$[t_j, t_{j+1}] \quad j = 0, \ldots, p$$

f satisfies the assumptions of Lemma 2.1 and so there exists $\eta_j \in (t_j, t_{j+1})$ $(j = 0, \ldots, p)$ such that

$$f(\eta_j) Lf(\eta_j) > 0. \tag{2.3}$$

If t_j is a zero point of the first order then $f(\eta_{j-1}) f(\eta_j) < 0$ and, according to (2.3) we get

$$Lf(\eta_{j-1}) Lf(\eta_j) < 0.$$

It follows that Lf is equal to zero at least one point of the interval (η_{j-1}, η_j).

If the order of t_j is greater than one we have according to Lemma 2.2 $f(t)\, Lf(t) < 0$ in the same neighbourhood of t_j. By (2.3) we get that Lf has at least two zero points in the interval (η_{j-1}, η_j).

If a is a zero point of order s $(s > 1)$ then by Lemma 2.2 $f(t)\, Lf(t) < 0$ in some right neighbourhood of a. It follows by (2.3) that Lf has a zero point in the interval (a, η_0).

If b is a zero point of the order s $(s > 1)$ we similarly obtain that Lf has a zero point in (η_p, b). And so

$$Z(Lf) \geqslant p + k .$$

Throughout this paper we shall use the following

LEMMA 2.4. *For every* $f \in \mathcal{M}$ $Z(Lf) \geqslant Z(f)$.

Proof. Consider two cases

Case 1. $Z(f) < \infty$. Since $Z(f) < \infty$ the sets $M_1(f)$, $M_2(f)$ are finite. Denote by $[\alpha_j, \beta_j]$, $j = 1, \ldots, r$ the intervals of $M_2(f)$ and arrange them so that $\alpha_j < \beta_j < \alpha_{j+1}$ for $j = 1, \ldots, r-1$.

Put

$$\beta_0 = \inf\{x \in (a, \alpha_1)\colon f \text{ is not identical to zero in any open subinterval of } (x, \alpha_1)\}.$$

$$\alpha_{r+1} = \sup\{x \in (\beta_r, b)\colon f \text{ is not identical to zero in any open subinterval of } (\beta_r, x)\}.$$

Since f is of class C^2 in $[\beta_j, \alpha_{j+1}]$, it satisfies the boundary conditions (2.1) and does not vanish identically there exists $t_j \in (\beta_j, \alpha_{j+1})$, $j = 0, \ldots, r$, such that $Lf(t_j) \neq 0$. Consequently the power of $M_2(f)$ is not greater than that of $M_2(Lf)$.

To prove that the power of $M_1(f)$ is not greater that the of $M_1(Lf)$ denote by $t_1, \ldots, t_s$ $(t_1 < t_2 < \ldots < t_s)$ the zero points of in (β_j, β_{j+1}) and put $t_0 = \beta_j$, $t_{s+1} = \alpha_{j+1}$ (j is fixed).

By Lemma 2.1 applied to $[t_i, t_{i+1}]$ there exists $\eta_i \in (t_i, t_{i+1})$ $(i = 0, \ldots, s-1)$ such that $f(\eta_i) Lf(\eta_i) > 0$.

Considering the following cases

1° $f(t_i) = 0,\ f'(t_i) \neq 0$

2° $f(t_i) = f'(t_i) = 0,\ f''(t_i) \neq 0$

3° $f(t_i) = f'(t_i) = f''(t_i) = 0$

and applying Lemma 2.2 in case 2° we get that Lf has at least one zero point in each interval (η_i, η_{i+1}) $(i = 0, \ldots, s-1)$.

By the same reasoning for $j = 0, \ldots, r$ we get that the power of $M_1(f)$ is not smaller than that of $M_1(Lf)$ which finishes the proof for the case $L(f) < \infty$.

Case 2° $Z(f) = \infty$. Suppose $Z(Lf) = n < \infty$ and choose $(n+1)$ intervals these can be one point intervals $[\alpha_1, \beta_1], \ldots, [\alpha_{n+1}, \beta_{n+1}]$ $(\alpha_1 < \beta_1 < \ldots < \beta_{n+1})$ from the set $M_1(f) \cup M_2(f)$. Since $Z(f) = \infty$ the above construction is possible and put $\beta_0 = a$, $\alpha_{n+2} = b$. Function f satisfies the boundary conditions (2.1) in $[\beta_j, \alpha_{j+1}]$ and does not vanish identically and so there exist $t_j \in (\beta_j, \alpha_{j+1})$ such that $f(t_j) \neq 0$ for $j = 0, \ldots, p+1$.

Denote by

$$x_j = \inf\{x \in (a, t_j)\colon f(t) \neq 0 \text{ for } t \in (x, t_j)\},$$

$$y_j = \sup\{x \in (t_j, b)\colon f(t) \neq 0 \text{ for } t \in (t_j, x)\},\ j = 0, \ldots, n+1.$$

Obviously $(x_j, y_j) \subset (\beta_j, \alpha_{j+1})$ and since f satisfies the assumptions of (2.1) in $[x_j, y_j]$ there exist $\eta_j \in (x_j, y_j)$ such that $f(\eta_j)Lf(\eta_j) > 0$, $j = 0, \ldots, n+1$.

We shall now prove that Lf has a zero point in (η_j, η_{j+1}). If $\alpha_{j+1} < \beta_{j+1}$ then Lf vanishes in $[\alpha_{j+1}, \beta_{j+1}/$.

If $\alpha_{j+1} = \beta_{j+1}$ then α_{j+1} is either a limit point of zeros of f and then $Lf(\alpha_{j+1}) = 0$ or α_{j+1} is an isolated zero point of f.

Suppose α_{j+1} is an isolated zero point of f. Put

$$\bar{\alpha}_{j+1} = \inf\{x \in (a, \alpha_{j+1})\colon f(t) \neq 0 \text{ for } t \in (x, \alpha_{j+1})\},$$

$$\bar{\beta}_{j+1} = \sup\{x \in (\alpha_{j+1}, b)\colon f(t) \neq 0 \text{ for } t \in (\alpha_{j+1}, x)\}.$$

Applying Lemma 2.1 to the intervals $[\bar{\alpha}_{j+1}, \alpha_{j+1}]$, $[\alpha_{j+1}, \bar{\beta}_{j+1}]$ we see that there exists $\eta_j \in (\bar{\alpha}_{j+1}, \alpha_{j+1})$, $\bar{\bar{\eta}}_{j+1} \in (\alpha_{j+1}, \bar{\beta}_{j+1})$ such that $f(\bar{\eta}_j)Lf(\bar{\eta}_j) > 0$ and

$$f(\bar{\bar{\eta}}_{j+1})Lf(\bar{\bar{\eta}}_{j+1}) > 0.$$

If $\bar{\alpha}_{j+1} < \eta_j$ then put $\bar{\eta}_j = \eta_j$ and if $\bar{\beta}_{j+1} > \eta_{j+1}$ put $\bar{\bar{\eta}}_{j+1} = \eta_{j+1}$. We have then $(\eta_j, \bar{\bar{\eta}}_{j+1}) \subset (\eta_j, \eta_{j+1})$. Considering the cases 1°, 2°, 3° as for $Z(f) < \infty$ we get that in the intervals $(\bar{\eta}_j, \bar{\bar{\eta}}_{j+1})$ and consequently in (η_j, η_{j+1}), $j = 0, \ldots, n+1$, there exists at least zero point of Lf. Since $Lf(\eta_j) \neq 0$ then $Z(Lf) \geqslant n+1$, which contradicts our assumption.

3. Composition of operators of the Sturm-Liouville type. Let $\mathcal{M}_j$, $j = 1, \ldots, k$ stand for the set of functions of class C^2 in $[a, b]$ which satisfy the following boundary conditions

$$\begin{aligned} &\alpha_1^{(j)}u(a) - \alpha_2^{(j)}u'(a) = 0 \\ &\beta_1^{(j)}u(b) + \beta_2^{(j)}u'(b) = 0 \end{aligned} \qquad \text{for } j = 1, \ldots, k \tag{3.1}$$

where $\alpha_1^{(j)} > 0$, $\beta_1^{(j)} > 0$, $\alpha_2^{(j)} \geqslant 0$, $\beta_2^{(j)} \geqslant 0$.

$\mathcal{M}_j$ are dense linear subspaces of L^2 ($[a, b]$).

Operators

$$L_j\colon \mathcal{M}_j \to L^2([a, b])$$

defined by the formula

$$L_j u(t) = -(p_j(t)u(t))' + q_j(t)u(t),$$

where $p_j(t) > 0$, $q_j(t) \geqslant 0$ for $t \in [a, b] p_j \in C^{2j-1}([a, b])$, $q_j \in C^{2j-2}([a, b])$, $j = 1, \ldots, k$, are linear and inversible operators of the Sturm-Liouville type.

$K_j = L_j^{-1}$ is an integral operator of the Fredholm type and let $G_j(x, t)$ stand for its kernel.

In what follows we shall consider the eigenfunctions and eigenvalues of operator

$$L = L_1 L_2 \dots L_k, \quad k \geqslant 1 .$$

Definition. The real number λ is called the eigenvalue of L if there exists $u \neq 0$ from the domain of D_L of the operator L such that $Lu = \lambda u$. The function u is then called an eigenfunction of L corresponding to the eigenvalue λ.

Since the equation $Lu = 0$ $(u \in D_L)$ has only a zero solution so $\lambda = 0$ is not an eigenvalue of L. Moreover every eigenfunction has at most a finite number of zero points in (a, b). Indeed, suppose there exists $t_0 \in [a, b]$ which is a limit point for the zeros of a certain eigenfunction u of an order equal to the class of u. And so the order of t_0 is greater or equal to $2k$ i.e.

$$u(t_0) = u'(t_0) = \dots = u^{(2k-1)}(t_0) = 0 .$$

By the uniqueness theorem for the Cauchy problem for the equation $Lu - \lambda u = 0$, $u = 0$, which contradicts the definition of the eigenfunction of L.

We shall prove the following

THEOREM 3.1. *The zero points of the eigenfunction u of the operator L are of the first order.*

Proof. Denote by $t_1, \dots, t_p$ $(a = t_0 < t_1 < \dots < t_p < t_{p+1} = b)$ the zero points of u. Suppose the order of zero point t_j is greater than one. By Lemma 2.3 $Z(L_k u) \geqslant p+1$. Applying Lemma 2.3 to the function $L_k u$ $(k-1)$ times we get

$$Z(Lu) \geqslant p+1 .$$

On the other hand $Lu = \lambda u$ and so $Z(Lu) = p$ which contradicts the previous inequality.

The same as for $k = 1$ we get the following

THEOREM 3.2. *Any two eigenfunctions of operator L corresponding to the same eigenvalue are linearly dependent.*

Proof. Suppose there exist eigenfunctions u and v of L corresponding to the eigenvalue λ which are linearly independent. By (3.1)

$$\alpha_1^{(k)} u(a) - \beta_2^{(k)} u'(a) = 0$$

$$\alpha_1^{(k)} v(a) - \alpha_2^{(k)} v'(a) = 0 ,$$

where $\alpha_1^{(k)} > 0$, $\alpha_2^{(k)} \geqslant 0$.
If $\alpha_2^{(k)} = 0$ then $u(a) = v(a) = 0$ and by theorem 3.1 $u'(a)v'(a) \neq 0$. Consequently there exist $\delta \in R$ such that $u'(a) - \delta v'(a) = 0$. But since $w = u - \delta v$ is an eigenfunction and $w(a) = w'(a) = 0$ we get $w \equiv 0$ (by 3.1). If $\alpha_2^k > 0$ we get $u(a)u'(a) \neq 0$ for every eigenfunction u. The function $w = u - \delta v$, where $\delta = \dfrac{u(a)}{v(a)}$ would be an eigenfunction and $w(a) = 0$ which is impossible for

$\alpha_2^{(k)} > 0$. We know that for $k = 1$ any two eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2([a, b])$.

It is chiefly because of this fact that we can define eigenfunctions and eigenvalues by variation methods. The following problem arises: are eigenfunctions corresponding to different eigenvalues orthogonal also for the case $k \geqslant 1$? To answer this question, note first that the functional $(\cdot, \cdot)$: $\mathcal{M}_k \times \mathcal{M}_k \to R$ defined by the formula

$$(\varphi, \psi)_k = \int_a^b \varphi(t) L_k \psi(t)\, dt$$

is a scalar product in $\mathcal{M}_k$.

We have the following

THEOREM 3.3. *If* $K = K_{k-2} \ldots K_1$ *is a symmetric operator in* $L^2([a, b])$ *then any eigenfunctions* u, v *of the operator* $L = L_1 \ldots L_k$ *corresponding to the eigenvalues* λ *and* μ *are orthogonal in* $\mathcal{M}_k$ *i.e.* $(u, v)_K = 0$. *Moreover the functions* u, Kv *are orthogonal in* $L^2([a, b])$.

Proof. Since u, v are eigenfunctions we have

$$L_k u = \lambda K u$$

$$L_k v = \mu K v\ .$$

Multiplying the first equality by v, the second by u integrating both formulas and substracting we get

$$0 = (v, u)_k - (u, v)_k = (\lambda - \mu)(u, Kv)_{L^2}\ .$$

Since $\lambda \neq \mu$, $(u, Kv)_{L^2} = 0$.

On the other hand

$$0 = \mu(u, Kv)_{L^2} = (u, L_k v)_{L^2} = (u, v)_k\ .$$

Operators of the form

$$K_1^{k-1},\ K_1 K_2 \ldots K_{p-1} K_p K_{p-1} \ldots K_2 K_1,\ K_1 K_2 K_1 K_3 K_4 K_3 K_1 K_2 K_1$$

are symmetric in $L^2([a, b])$ and positive i.e. $(K\varphi, \varphi)_{L^2} > 0$ for $\varphi \in L^2$, $\varphi \neq 0$.

Suppose $K = K_{k-1} \ldots K_1$ is a symmetric and positive operator. Then the functional $[\cdot, \cdot]_K$ given by the formula

$$[u, v]_K = (Kv, u)_{L^2}$$

forms a scalar product in $L^2([a, b])$. Since

$$\|\varphi\|_K^2 \leqslant \|K\|\, \|\varphi\|_{L^2}^2\ ,$$

where $\|\varphi\|_K^2 = [\varphi, \varphi]_K$ and $K^{-1} = L_{k-1} \ldots L_1$ is unbounded, the space L^2 with a scalar product $[\cdot, \cdot]_K$ is not complete. Space L^2 can be extended to a complete space (see [1] p. 303) which we denote by L_K^2.

Using the above notations Theorem 3.3 takes the from

THEOREM 3.3a. *If* $K = K_{k-1} \dots K_1$ *is a symmetric and positive operator in* L^2 *then any two eigenfunctions of the operator* L *corresponding to different eigenvalues are orthogonal in* L^2_K.

4. Eigenvalues and eigenfunctions of the operator $K_k K$. Suppose $K = K_{k-1} \dots K_1$ to be symmetric and positive in L^2. We shall prove the following

THEOREM 4.1. *The operator* $K_k K$ *is bounded, symmetric, compact and normal in* L^2_K.

Proof. 1° Boundedness. To prove that $K_k K$ is bounded we shall use the following

LEMMA 4.1. ([8], p. 287). *For every positive and symmetric operator* A *there exists exactly one positive and symmetric operator* $A^{\frac{1}{2}}$ *such that* $(A^{\frac{1}{2}})^2 = A$ *which commutes with every operator commuting with* A.

We shall prove that $K_k K$ is bounded in L^2_K. Let $u \in L^2([a, b])$, $v = K^{\frac{1}{2}} u$.

$$\begin{aligned} \|K_k K_u\|^2_K &= (K K_k K u, K_k K u)_{L^2} \leqslant \|K\|_{L^2} (K_k K u, K_k K u)_{L^2} \\ &= \|K\|_{L^2} (K_k K_k K u, K u)_{L2} \leqslant \|K\|_{L^2} \|K^2_k\|_{L^2} (K u, K u)_{L^2} \\ &= C (K K^{\frac{1}{2}} u, K^{\frac{1}{2}} u)_{L^2} \leqslant C \|K\|_{L^2} (K^{\frac{1}{2}} u, K^{\frac{1}{2}} u)_{L^2} = M \|u\|^2_K \end{aligned}$$

where $C = \|K\|_{L^2} \|K^2_k\|_{L^2}$, $M = C \|K\|_{L^2}$.

2° Symmetry. Let $u, v \in L^2([a, b])$.

$$[K_k K u, v]_K = (K_k K u, K v)_{L^2} = (K u, K_k K v)_{L^2} = [u, K_k K v]_K .$$

Since L^2 is dense in L^2_K for every $\varphi, \psi \in L^2_K$ there exist sequences

$$u_n, v_n, n = 1, 2, \dots, u_n, v_n \in L^2$$

such that $\lim_{n\to\infty} u_n = \varphi$, $\lim_{n\to\infty} v_n = \psi$, where the convergence is in norm of L^2_K. Since both operator and scalar product are continuous we have

$$[K_k K \varphi, \psi]_K = \lim_{n\to\infty} [K_k K u_n, v_n]_K = \lim_{n\to\infty} [u_n, K_k K v_n]_K = [\varphi, K_k K]_K .$$

3° The operator $K_k K$ is compact. This fact may be proved in the same way as in [7] p. 208.

4° Proof that $K_k K$ is a normal operator. The operator $K_k K$ is self-adjoint since it is both continuous and symmetric in L^2_K. Since every self-adjoint operator is normal ([7] p. 158) we have 4°.

Because $K_k K$ is a self-adjoint, positive and compact operator in L^2_K we can use the well-known variation method to obtain its eigenvalues and eigenfunctions.

The first eigenvalue χ_1 equals the maximum value of the functional

$$H(u) = \frac{[K_k K u, u]_K}{[u, u]_K} .$$