ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCCLXXV PRACE MATEMATYCZNE, ZESZYT 19 1977

On the asymptotically solvable linear boundary value problem

by J. PIÓREK

The purpose of this paper is to prove a theorem on the so-called asymptotically solvable linear boundary value problem.

The first part of the paper contains some auxiliary results on the properties of topological function spaces.

In the second part we present the problem and state our main result, Theorem 1 on the Baire category of the set of functions for which some special linear boundary value problem is asymptotically solvable.

In the third part we prove the lemma which gives us a criterion of the existence of a solution of this problem.

The fourth part is devoted to the proof of Theorem 1.

In the last part we give an example showing that the condition of the asymptotical solvability appearing in Theorem 1 is essential, i.e. that it cannot be changed by the condition of solvability of the problem.

In this paper we use some former results similar to Theorem 1, namely the results given in the paper [1] of Lasota and Yorke and in the author's paper [2].

1. Preliminary notes. Suppose that X is a topological space and $A \subset X$.

Definition. A is nowhere dense in X iff $\operatorname{int} \bar{A} = \emptyset$. A is a set of the first category in X iff A is a countable union of nowhere dense sets in X. A is a generic set in X iff $X \setminus A$ is a set of the first category in X.

In the following lemmas we gather some properties of generic sets.

LEMMA 1. *Let X be a topological space and Y its subspace. Let Y be generic in X. If a set $A \subset Y$ is generic in Y, then A is generic in X.*

The proof of Lemma 1 is simple and it follows immediately from the definition of a generic set and from the fact that a nowhere dense set in Y is also nowhere dense in a larger set X.

LEMMA 2. *Let X be a metric space and let $X_1 \subset X$ be dense in X. Let a real function φ defined on X be such that*

$$\varphi(x_n) \to 0 \quad \text{when} \quad x_n \to x \quad \text{for} \quad x \in X_1, x_n \in X .$$

Let us put $A = \{x \in X\colon \varphi(x) \neq 0\}$.
Then the set $X \setminus A$ is generic.

A simple proof of this Lemma may be found, for example, in [2] and is therefore omitted here.

Now we shall consider some properties of function spaces which appear in problems of differential equations in Banach space.

Let $(E, \|\cdot\|)$ be a Banach space. Let $U = [0,1]\times E$. Let $E_n = \{x \in E: \|x\| \leqslant n\}$ and $U_n = [0,1]\times E_n$. Notice that the sets E_n are closed and bounded in E and compose a sequence increasing to E. Similarly, sets U_n are closed and bounded in U, and compose a sequence increasing to U and a basis of bounded sets in U; that is to say, for every bounded set $B \subset U$ there is a positive integer n such that $B \subset U_n$. $\mathfrak{X}$ will denote the set of all continuous functions f: $U \to E$. We endow the set $\mathfrak{X}$ with the topology of the uniform convergence on bounded sets. In this topology, the basis of neighbourhoods is a family of sets of the form

$$N(f, n, \varepsilon) = \{g \in \mathfrak{X}: \sup_{u\in U_n} \|f(u)-g(u)\| < n\} \quad \text{for} \quad f \in \mathfrak{X},\ n = 1, 2, \ldots$$

and $\varepsilon > 0$.

This topology is metrizable and — as is easy to see — consistent with the topology given in $\mathfrak{X}$ by the bounded metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\sup\{\|f(u)-g(u)\|:\ u \in U_n\}}{1+\sup\{\|f(u)-g(u)\|:\ u \in U_n\}} \quad \text{for } f, g \in \mathfrak{X}\,.$$

$\mathfrak{B}$ will denote the topological subspace of $\mathfrak{X}$ consisting of bounded functions $f \in \mathfrak{X}$. That is to say, $f \in \mathfrak{B}$ iff $f \in \mathfrak{X}$ and the set $f(U)$ is bounded in E.

Definition. Function $f \in \mathfrak{X}$ is called locally Lipschitzean (more precisely: locally Lipschitzean with respect to the variable in E) iff for every point $u_0 \in U$ there exists a neighbourhood V of this point and a positive number L such that $\|f(t, x)-f(t, y)\| \leqslant L\|x-y\|$ for $(t, x), (t, y) \in V$ where $t \in [0,1]$ and $x, y \in E$.

The following lemma on locally Lipschitzean functions will be useful.

LEMMA 3. *The set of functions $f \in \mathfrak{X}$ which are locally Lipschitzean is a dense set in the space $\mathfrak{X}$.*

Lemma 3 follows for example from Lemma 1 in [1].

From the proof of this lemma follows

LEMMA 3′. *The set of functions $f \in \mathfrak{B}$ which are locally Lipschitzean is a dense set in the space $\mathfrak{B}$.*

C [0,1] will denote the Banach space of continuous functions x: $[0,1]\to E$ with sup norm.

All derivatives and integrals which appear in this paper should be interpreted in the strong sense, i.e. as limits in the norm of a Banach space E.

2. Presentation of the problem. Let us consider the Cauchy problem

$$\begin{cases} x' = f(t, x) & (1)\\ x(0) = r & (2)\end{cases}$$

where $f \in \mathfrak{X}$ and $r \in E$.

Definition. The solution $x\colon \Delta \to E$ of the problem (1)–(2) is called an unlimited solution iff either $\Delta = [0,1]$ or $\Delta = [0,a)$ with $a \leqslant 1$ and the limit $\lim_{t\to a} x(t)$ does not exist.

If function f in (1) is bounded then each unlimited solution of (1)–(2) is defined on the whole interval $[0,1]$. Indeed, if $x\colon \Delta \to E$ is such a solution, then

$$\|x(t)\| \leqslant \int_0^t \|f(s, x(s))\| ds \leqslant Mt \leqslant M \quad \text{for} \quad t \in \Delta$$

and

$$\|x(t_1) - x(t_2)\| \leqslant \left| \int_{t_1}^{t_2} \|f(s, x(s))\| ds \leqslant M |t_1 - t_2| \right| \quad \text{for} \quad t_1, t_2 \in \Delta$$

where

$$M = \sup\{\|f(u)\|\colon\ u \in U\} < +\infty .$$

Thus $x\colon \Delta \to E$ is a bounded Lipschitzean function. Being unlimited it must be defined on $[0,1]$.

Let us fix point $r \in E$. Let us denote by $\mathfrak{B}_0$ the set of such bounded functions $f \in \mathfrak{B}$ for which the problem (1)–(2) has exactly one unlimited solution. By virtue of Theorem 1 in [1] or Theorem 1 in [2] the set $\mathfrak{B}_0$ is generic in $\mathfrak{B}$. Let $L\colon C[0,1] \to E$ be a linear continuous operator. We define function $\varphi\colon \mathfrak{B}_0 \to \boldsymbol{R}$ setting

$$\varphi(f) = \inf\{\|Lx - r\|\colon\ x' = f(t, x)\} \quad \text{for} \quad f \in \mathfrak{B}_0,\ x \in C[0,1] .$$

Now we may introduce the following definition proposed by A. Lasota.

Definition. The linear boundary value problem

$$\begin{cases} x' = f(t, x) & (1) \\ Lx = r & (3) \end{cases}$$

is asymptotically solvable iff $\varphi(f) = 0$.

Our main theorem touches the special kind of operator L, namely an operator of the form

$$Lx = Ax(0) + Bx(1) \quad \text{for} \quad x \in C[0,1] ,$$

where operator $A\colon E \to E$ is linear, bounded and invertible, operator $B\colon E \to E$ is linear, bounded and compact and $\|A^{-1}B\| < 1$. The main theorem is as follows.

THEOREM 1. *Let $r \in E$ be fixed and let $L\colon C[0,1] \to E$ be an operator of the form described above. Then, in the topological space $\mathfrak{B}$, the set of functions $f \in \mathfrak{B}$ for which the problem*

$$\begin{cases} x' = f(t, x) & (1) \\ Lx = Ax(0) + Bx(1) = r & (4) \end{cases}$$

is asymptotically solvable is generic.

3. Existence lemma. LEMMA 4. *Let $f \in \mathfrak{B}$ be a locally Lipschitzean function such that*

$$\sup\{\|f(u)\|\colon\ u \in U\} = M < \infty .$$

Let $L\colon C[0,1] \to E$ be of the form $Lx = Ax(0) + Bx(1)$ where $A\colon E \to E$ is linear, bounded and invertible, $B\colon E \to E$ is linear, bounded and compact and $\|A^{-1}B\| < 1$, $\|A\| \geqslant m > 0$. Then, for every point $r \in E$ there exists at least one solution of the problem (1)–(4).

Proof of the lemma. Let us consider the Cauchy problem

$$\begin{cases} x' = f(t, x) & (1)\\ x(0) = c \quad \text{for} \quad c \in E . & (5)\end{cases}$$

From Lemma 2 in [1] it follows that problem (1)–(5) has a unique unlimited solution; let us denote it by x_c. Since $f \in \mathfrak{B}$ this solution is defined on interval $[0,1]$ (see point 2, above). Thus we may define correctly the operator $D\colon E \to E$ such that for every $c \in E$ $D(c) = x_c(1)$, i.e. $D(c)$ is a value of the solution of the problem (1)–(5) in the point $t = 1$. Since, by assumption, f is bounded and locally Lipschitzean, operator D is continuous and bounded. (This follows from Lemma 2 in [1].)

Let $R\colon E \to E$ denote a bounded homeomorphism given by the formula

$$R\colon x \to r - x \quad \text{for} \quad x \in E .$$

Now the existence of a solution of the problem (1)–(4) is equivalent to the existence of a fixed point of the operator $F\colon E \to E$ defined by the equality

$$F = A^{-1}\cdot R\cdot B\cdot D .$$

It is easy to see that F is a continuous compact operator on the space E. To prove the existence of a fixed point it suffices to apply Schauder's Fixed Point Theorem, i.e. it suffices to find a nonempty closed, convex, bounded set $K \subset E$ such that $F(K) \subset K$. Let us set

$$K = K(a) = \{c \in E\colon\ \|c - r\| \leqslant a\} \quad \text{for} \quad a > 0 .$$

We shall see that for a sufficiently great a the set $K(a)$ has the desired properties. Of course, for every $a > 0$ the set $K(a)$ (ball in E) is nonempty, closed, convex and bounded. So it remains to choose a number $a > 0$ such that $F(K) \subset K$.

Let $c \in K$. Then

$$\|Dc - c\| = \|x_c(1) - c\| \leqslant \int_0^1 \|f(s, x_c(s))\|\, ds \leqslant M$$

and consequently

$$\|Dc\| \leqslant \|c\| + M,\ \|(BD)c\| \leqslant \|B\|\cdot\|Dc\| \leqslant \|B\|(\|c\| + M) .$$

Further

$$\|(RBD)c\| = \|r-(BD)c\| \leqslant \|r\| + \|B\|(\|c\|+M) .$$

Thus

$$\|Fc\| \leqslant \|A^{-1}\|\cdot\|(RBD)c\| \leqslant \|A^{-1}\|[\|r\|+\|B\|(\|c\|+M)] .$$

Finally

$$\|Fc-r\| \leqslant \|Fc\|+\|r\|$$

and a simple computation shows that $\|Fc-r\| \leqslant a$ (i.e. $Fc \in K$) if

$$a \geqslant \left\{\|r\|+\frac{1}{m}[\|r\|+\|B\|(\|r\|+M)]\right\}(1-\|A^{-1}B\|)^{-1} .$$

The lemma is proved.

4. Proof of theorem 1. Let $\mathfrak{B}_1$ be a set of functions $f \in \mathfrak{B}_0$ and locally Lipschitzean. Set $\mathfrak{B}_1$ is dense in $\mathfrak{B}_0$. (For definition of $\mathfrak{B}_0$ see point 2.) From Lemma 4 it follows that on the set $\mathfrak{B}_1$ problem (1)–(4) is solvable. Thus, $\varphi(f) = 0$ for $f \in \mathfrak{B}_1$. We shall now prove that

$$\varphi(g)\to 0 \quad \text{when} \quad g\to f \text{ in } \mathfrak{X} \quad \text{for} \quad g \in \mathfrak{B}_0, f \in \mathfrak{B}_1 , \tag{6}$$

in other words, that

$$\forall_{\varepsilon>0}\exists_{\delta>0} d(f,g) < \delta \Rightarrow \varphi(g) < \varepsilon .$$

Since operator L is continuous, we have

$$\forall_{\varepsilon>0}\exists_{\eta>0}\|x_f-x_g\| < \eta \Rightarrow \|Lx_f-Lx_g\| = \|Lx_g-r\| < \varepsilon ,$$

where function x_f is a solution of the problem (1)–(4) for $f \in \mathfrak{B}_1$ and function x_g is an unlimited solution of the Cauchy problem

$$\begin{cases} x' = g(t,x) \\ x(0) = x_f(0) \end{cases}$$

for $g \in \mathfrak{B}_0$.

Since the solution depends continuously on right-hand side function (see Lemma 2 in [2]) we have

$$\forall_{\eta>0}\exists_{\delta>0} d(f,g) < \delta \Rightarrow \|x_f-x_g\| < \eta .$$

The desired property (6) follows immediately from last two implications.

So we have proved that function φ satisfies all the assumptions of Lemma 2 with $X = \mathfrak{B}_0$ and $X_1 = \mathfrak{B}_1$. So, by virtue of this lemma, the set $\mathfrak{B}^*$ of functions $f \in \mathfrak{B}_0$ for which the problem (1)–(4) is asymptotically solvable is a generic set in $\mathfrak{B}_0$. But we have observed (see point 2) that set $\mathfrak{B}_0$ is generic in $\mathfrak{B}$. Then from Lemma 1 it follows that $\mathfrak{B}^*$ is a generic set in space $\mathfrak{B}$. This finishes the proof of Theorem 1.

5. Example. Let $E = L^2[0, \infty)$ be a Hilbert space of square integrable functions $x\colon [0, \infty) \to \mathbf{R}$ with the inner product

$$\subset x, y \supset = \int_0^\infty x(s)y(s)\,ds \quad \text{for} \quad x, y \in E\,.$$

Let us consider the function $f\colon \mathbf{R}\times E \to E$ given by the formula

$$f(t, x) = 0 \quad \text{for} \quad t \leqslant 0$$

$$f(t, x)(s) = 1_{[0,t]}(s)\left(\frac{|x(s)|}{\sqrt{\|x\|}} + \max\{0, t^2/4 - \|x\|\}\right) \quad \text{for} \quad t > 0\,.$$

In paper [1] the authors proved that this function is continuous and that for this function the Cauchy problem

$$\begin{cases} x' = f(t, x) \\ x(0) = 0 \end{cases}$$

has no solution. This means that the linear boundary value problem

$$\begin{cases} x' = f(t, x) \\ Lx = 0 \end{cases} \tag{7}$$

where $Lx = x(0)$ is unsolvable. On the other hand, we shall prove that function f is Lipschitzean in some neighbourhood of each point $(t, x) \in \mathbf{R}\times E$ such that $x \neq 0$. But then, by virtue of Lemma 2 in [2], for each $r \in E$ and $r \neq 0$ there exists a solution of the problem

$$\begin{cases} x' = f(t, x) \\ Lx = r. \end{cases}$$

This means that problem (7) is asymptotically solvable. To prove that the function f is locally Lipschitzean let us fix the point $(t_0, x_0) \in \mathbf{R}\times E$ such that $x_0 \neq 0$ and let us fix a neighbourhood V of this point:

$$V = \{(t, x) \in \mathbf{R}\times E\colon\ |t - t_0| < \varepsilon,\ \|x - x_0\| < \varepsilon\}\,,$$

such that $\|x_0\| > \varepsilon$. Let us choose two points (t, x) and (t, y) in V and let us estimate

$$\|f(t, x) - f(t, y)\|^2$$

$$= \int_0^t \left(\frac{|x(s)|}{\sqrt{\|x\|}} - \frac{|y(s)|}{\sqrt{\|y\|}} + \max\{0, t^2/4 - \|x\|\} - \max\{0, t^2/4 - \|y\|\}\right)^2 ds$$

$$\leqslant 2\int_0^t \left(\frac{|x(s)|}{\sqrt{\|x\|}} - \frac{|y(s)|}{\sqrt{\|y\|}}\right)^2 ds + 2\int_0^t (\max\{0, t^2/4 - \|x\| - \max\{0, t^2/4 - \|y\|\})^2 ds\,.$$

Let us denote the components of the last sum by J_1 and J_2. Then

$$J_1 = \frac{2}{\|x\|\cdot\|y\|}\int_0^t \big(\sqrt{\|y\|}\,|x(s)| - \sqrt{\|x\|}\,|y(s)|\big)^2 ds$$

$$\leqslant \frac{4}{\|x\|\cdot\|y\|}\left[\|y\|\int_0^t (|x(s)|-|y(s)|)^2 ds + \|y\|^2\left(\frac{\|y\|-\|x\|}{\sqrt{\|x\|}+\sqrt{\|y\|}}\right)^2\right]$$

$$\leqslant \frac{4}{\|x\|}\left[\int_0^\infty |x(s)-y(s)|^2 ds + \frac{\|y\|}{(\sqrt{\|x\|}+\sqrt{\|y\|})^2}\|x-y\|^2\right]$$

$$= \frac{4}{\|x\|}\left(1+\frac{\|y\|}{(\sqrt{\|x\|}+\sqrt{\|y\|})^2}\right)\|x-y\|^2 .$$

To estimate J_2 let us fix, for example, that $\|x\| > \|y\|$ (if $\|x\| = \|y\|$ then $J_2 = 0$). Then

$$J_2 = \begin{cases} 2t\cdot 0 = 0, & t^2/4 \leqslant \|y\| \\ 2t(t^2/4-\|y\|)^2 \leqslant 2t(\|x\|-\|y\|)^2 \leqslant 2t\|x-y\|^2, & \|y\| \leqslant t^2/4 \leqslant \|x\| \\ 2t(\|x\|-\|y\|)^2 \leqslant 2t\|x-y\|^2, & \|x\| \leqslant t^2/4 . \end{cases}$$

Thus in any case $J_2 \leqslant 2t\|x-y\|^2$.
Now a simple computation shows that

$$\|f(t,x)-f(t,y\| \leqslant L\|x-y\|, \quad \text{for} \quad (t,x),(t,y)\in V ,$$

where

$$L^2 = \frac{4}{\|x_0\|-\varepsilon}\left(1+\frac{\|x_0\|+\varepsilon}{4(\|x_0\|-\varepsilon)}\right)+2t_0+2\varepsilon .$$

References

[1] A. Lasota and J. A. Yorke, *The generic property of existence of solutions of differential equations in Banach space,* J. Differential Equations 13 (1973), 1—12.

[2] J. Piórek, *On integral equations in Banach space,* to appear in Ann. Pol. Math.