

Analytic continuation of series of homogeneous polynomials in topological vector spaces

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1. Introduction. Let X, Y be topological vector spaces over the field of complex numbers \mathbb{C} . Moreover let X be a Fréchet space, Y — locally convex and sequentially complete with topology determined by a filtrant set of seminorms $\mathfrak{G}(Y)$. Let $\mathfrak{P}^n(X, Y)$ denote the vector space of all continuous homogeneous polynomials of X into Y of degree n ($\mathfrak{P}^0(X, Y) = Y$).

We shall consider a series of the form

$$(1.1) \quad f(x) = \sum_{n=0}^{\infty} f_n(x), \quad f_n \in \mathfrak{P}^n(X, Y),$$

that converges in a neighbourhood of $0 \in X$. Then the function f , given by (1.1), is analytic in a balanced neighbourhood of $0 \in X$ (i.e. in a domain of convergence of series (1.1)).

Let G_f denote the Mittag-Leffler star of the function f , i.e. the largest starlike (with respect to zero) open set such that f can be analytically continued to G_f .

The aim of this paper is to find the analytic continuation of the function f to G_f . The paper extends the results of [6] to the case where \mathbb{C}^n, \mathbb{C} are replaced by X, Y respectively.

All definitions and theorems concerning analytic functions in topological vector spaces may be found in [3], [4].

2. Basic definitions. In the sequel X, Y will denote topological vector spaces over \mathbb{C} , Y being locally convex and sequentially complete with topology determined by a filtrant set of seminorms $\mathfrak{G}(Y)$. Let U be a non-empty open subset of X .

Definition 2.1. A continuous function $f: U \rightarrow Y$ is *analytic* in U if for every $x \in U$ there exists a series $\sum_{n=0}^{\infty} f_n$ ($f_n \in \mathfrak{P}^n(X, Y)$) such that

$$f(x+h) = \sum_{n=0}^{\infty} f_n(h)$$

for all h in a neighbourhood of $0 \in X$.

By $\mathcal{A}(U, Y)$ we denote the vector space of all analytic functions $f: U \rightarrow Y$. If $Y = \mathbb{C}$ we shall write O_U instead of $\mathcal{A}(U, \mathbb{C})$. If we say that function f is analytic we mean that it is analytic in its whole domain, unless it is said that f is analytic in a smaller set.

Definition 2.2. We say that a function $f \in \mathcal{A}(U, Y)$ is *continuable onto a domain* $U_1 \supset U$ if there exists a function $g \in \mathcal{A}(U_1, Y)$ such that $g|_U = f$.

Definition 2.3. A function $f: U \rightarrow [-\infty, \infty)$ is *plurisubharmonic* if 1) f is upper semicontinuous, 2) for every $u \in U, x \in X \setminus \{0\}$ the function $f_{u,x}: \lambda \rightarrow f(u + \lambda x)$ is subharmonic in $\{\lambda \in \mathbb{C}: u + \lambda x \in U\}$.

3. Separately analytic functions. Here we give some theorems concerning separately analytic functions which will be used in section 4.

Let us remind ourselves that a function $f: X_1 \times X_2 \supset D \rightarrow Y$ is called *separately analytic* if for every $(u_0, v_0) \in D$ the functions $f_{u_0}: v \rightarrow f(u_0, v)$ and $f_{v_0}: u \rightarrow f(u, v_0)$ are analytic in $\{v \in X_2: (u_0, v) \in D\}$ and $\{u \in X_1: (u, v_0) \in D\}$, respectively.

LEMMA 3.1. *Let U be a domain in a Baire space X . Let $\lambda_0 \in \mathbb{C}; \varepsilon, \varepsilon_1 \in (0, \infty), \varepsilon_1 < \varepsilon$. If a function $f: U \times B(\lambda_0, \varepsilon) \rightarrow Y$ is analytic in $U \times B(\lambda_0, \varepsilon_1)$ and if $f_x: B(\lambda_0, \varepsilon) \ni \lambda \rightarrow f(x, \lambda) \in Y$ is analytic for each $x \in U$ then f is analytic in $U \times B(\lambda_0, \varepsilon)$.*

Proof. Since $X \times \mathbb{C}$ is a Baire space ([14], th. 2) and since f is analytic in $U \times B(\lambda_0, \varepsilon_1)$ it is enough to show that f is analytic on affine lines ([3], th. 6.1). But a function of one complex variable is analytic iff it is weakly analytic; so we may assume that $Y = \mathbb{C}$.

Given two arbitrary points $x_1 \in U, x_2 \in X \setminus \{0\}$ put $A = \{\tau \in \mathbb{C}: x_1 + \tau x_2 \in U\}$. Fix x_1, x_2 and consider the following function of two complex variables

$$f^*: A \times B(\lambda_0, \varepsilon) \ni (\tau, \lambda) \rightarrow f(x_1 + \tau x_2, \lambda) \in \mathbb{C}.$$

By hypothesis f^* is analytic in $A \times B(\lambda_0, \varepsilon_1)$ and for each $\tau \in A$ $f_\tau^*: \lambda \rightarrow f^*(\tau, \lambda)$ is analytic in $B(\lambda_0, \varepsilon)$. Thus f^* is analytic in $A \times B(\lambda_0, \varepsilon)$ (see [5], th. 2, p. 139). Hence f is analytic on affine lines.

LEMMA 3.2. *Assume that $X_1 \times X_2$ is a Baire space. Let U be a domain in X_1 and let \hat{W}, \hat{V} be balanced neighbourhoods of zero in X_2 such that $\hat{W} \subset \hat{V}$. Put $V = v_0 + \hat{V}, W = v_0 + \hat{W}$, where v_0 is an arbitrary point in X_2 . If $f: U \times V \rightarrow Y$ is analytic in $U \times W$ and $f_x: V \ni v \rightarrow f(x, v)$ is analytic for every $x \in U$ then f is analytic in $U \times V$.*

Proof. It is enough to prove that $f_v: U \ni x \rightarrow f(x, v) \in Y$ is analytic for every $v \in V$ and then to apply the Hartogs Theorem for topological vector spaces ([3], Corollary 6.2.). (Here we do not need the assumption that one of the spaces X_j ($j = 1, 2$) is metrizable because f is analytic in $U \times W$.)

Fix $v \in V$. If $v = v_0$ then f_v is analytic in U by hypothesis. Suppose then that $v \neq v_0$ and write

$$\begin{aligned} A_V &= \{\lambda \in \mathbb{C}: \lambda(v - v_0) \in \hat{V}\} \\ A_W &= \{\lambda \in \mathbb{C}: \lambda(v - v_0) \in \hat{W}\} \end{aligned}$$