

Analytic continuation of series of homogeneous polynomials in topological vector spaces

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1. Introduction. Let X, Y be topological vector spaces over the field of complex numbers \mathbb{C} . Moreover let X be a Fréchet space, Y — locally convex and sequentially complete with topology determined by a filtrant set of seminorms $\mathfrak{G}(Y)$. Let $\mathfrak{P}^n(X, Y)$ denote the vector space of all continuous homogeneous polynomials of X into Y of degree n ($\mathfrak{P}^0(X, Y) = Y$).

We shall consider a series of the form

$$(1.1) \quad f(x) = \sum_{n=0}^{\infty} f_n(x), \quad f_n \in \mathfrak{P}^n(X, Y),$$

that converges in a neighbourhood of $0 \in X$. Then the function f , given by (1.1), is analytic in a balanced neighbourhood of $0 \in X$ (i.e. in a domain of convergence of series (1.1)).

Let G_f denote the Mittag-Leffler star of the function f , i.e. the largest starlike (with respect to zero) open set such that f can be analytically continued to G_f .

The aim of this paper is to find the analytic continuation of the function f to G_f . The paper extends the results of [6] to the case where \mathbb{C}^n, \mathbb{C} are replaced by X, Y respectively.

All definitions and theorems concerning analytic functions in topological vector spaces may be found in [3], [4].

2. Basic definitions. In the sequel X, Y will denote topological vector spaces over \mathbb{C} , Y being locally convex and sequentially complete with topology determined by a filtrant set of seminorms $\mathfrak{G}(Y)$. Let U be a non-empty open subset of X .

Definition 2.1. A continuous function $f: U \rightarrow Y$ is *analytic* in U if for every $x \in U$ there exists a series $\sum_{n=0}^{\infty} f_n$ ($f_n \in \mathfrak{P}^n(X, Y)$) such that

$$f(x+h) = \sum_{n=0}^{\infty} f_n(h)$$

for all h in a neighbourhood of $0 \in X$.

By $\mathcal{A}(U, Y)$ we denote the vector space of all analytic functions $f: U \rightarrow Y$. If $Y = \mathbb{C}$ we shall write O_U instead of $\mathcal{A}(U, \mathbb{C})$. If we say that function f is analytic we mean that it is analytic in its whole domain, unless it is said that f is analytic in a smaller set.

Definition 2.2. We say that a function $f \in \mathcal{A}(U, Y)$ is *continuable onto a domain* $U_1 \supset U$ if there exists a function $g \in \mathcal{A}(U_1, Y)$ such that $g|_U = f$.

Definition 2.3. A function $f: U \rightarrow [-\infty, \infty)$ is *plurisubharmonic* if 1) f is upper semicontinuous, 2) for every $u \in U, x \in X \setminus \{0\}$ the function $f_{u,x}: \lambda \rightarrow f(u + \lambda x)$ is subharmonic in $\{\lambda \in \mathbb{C}: u + \lambda x \in U\}$.

3. Separately analytic functions. Here we give some theorems concerning separately analytic functions which will be used in section 4.

Let us remind ourselves that a function $f: X_1 \times X_2 \supset D \rightarrow Y$ is called *separately analytic* if for every $(u_0, v_0) \in D$ the functions $f_{u_0}: v \rightarrow f(u_0, v)$ and $f_{v_0}: u \rightarrow f(u, v_0)$ are analytic in $\{v \in X_2: (u_0, v) \in D\}$ and $\{u \in X_1: (u, v_0) \in D\}$, respectively.

LEMMA 3.1. *Let U be a domain in a Baire space X . Let $\lambda_0 \in \mathbb{C}; \varepsilon, \varepsilon_1 \in (0, \infty), \varepsilon_1 < \varepsilon$. If a function $f: U \times B(\lambda_0, \varepsilon) \rightarrow Y$ is analytic in $U \times B(\lambda_0, \varepsilon_1)$ and if $f_x: B(\lambda_0, \varepsilon) \ni \lambda \rightarrow f(x, \lambda) \in Y$ is analytic for each $x \in U$ then f is analytic in $U \times B(\lambda_0, \varepsilon)$.*

Proof. Since $X \times \mathbb{C}$ is a Baire space ([14], th. 2) and since f is analytic in $U \times B(\lambda_0, \varepsilon_1)$ it is enough to show that f is analytic on affine lines ([3], th. 6.1). But a function of one complex variable is analytic iff it is weakly analytic; so we may assume that $Y = \mathbb{C}$.

Given two arbitrary points $x_1 \in U, x_2 \in X \setminus \{0\}$ put $A = \{\tau \in \mathbb{C}: x_1 + \tau x_2 \in U\}$. Fix x_1, x_2 and consider the following function of two complex variables

$$f^*: A \times B(\lambda_0, \varepsilon) \ni (\tau, \lambda) \rightarrow f(x_1 + \tau x_2, \lambda) \in \mathbb{C}.$$

By hypothesis f^* is analytic in $A \times B(\lambda_0, \varepsilon_1)$ and for each $\tau \in A$ $f_\tau^*: \lambda \rightarrow f^*(\tau, \lambda)$ is analytic in $B(\lambda_0, \varepsilon)$. Thus f^* is analytic in $A \times B(\lambda_0, \varepsilon)$ (see [5], th. 2, p. 139). Hence f is analytic on affine lines.

LEMMA 3.2. *Assume that $X_1 \times X_2$ is a Baire space. Let U be a domain in X_1 and let \hat{W}, \hat{V} be balanced neighbourhoods of zero in X_2 such that $\hat{W} \subset \hat{V}$. Put $V = v_0 + \hat{V}, W = v_0 + \hat{W}$, where v_0 is an arbitrary point in X_2 . If $f: U \times V \rightarrow Y$ is analytic in $U \times W$ and $f_x: V \ni v \rightarrow f(x, v)$ is analytic for every $x \in U$ then f is analytic in $U \times V$.*

Proof. It is enough to prove that $f_v: U \ni x \rightarrow f(x, v) \in Y$ is analytic for every $v \in V$ and then to apply the Hartogs Theorem for topological vector spaces ([3], Corollary 6.2.). (Here we do not need the assumption that one of the spaces X_j ($j = 1, 2$) is metrizable because f is analytic in $U \times W$.)

Fix $v \in V$. If $v = v_0$ then f_v is analytic in U by hypothesis. Suppose then that $v \neq v_0$ and write

$$A_V = \{\lambda \in \mathbb{C}: \lambda(v - v_0) \in \hat{V}\}$$

$$A_W = \{\lambda \in \mathbb{C}: \lambda(v - v_0) \in \hat{W}\}$$

Since \hat{V} and \hat{W} are balanced neighbourhoods of zero the sets A_V and A_W are discs in \mathbb{C} with centre in 0. Moreover $A_W \subset A_V$. The function $g: U \times A_V \ni (x, \lambda) \rightarrow f(x, v_0 + \lambda(v - v_0))$ satisfies the assumptions of Lemma 3.1., thus it is analytic in $U \times A_V$. Observe that $1 \in A_V$. Hence $g_1: U \ni x \rightarrow g(x, 1) \in Y$ is analytic in U . But $g_1 = f_v$, so the proof is concluded.

THEOREM 3.1. (*the generalized Hartogs Theorem*) Assume that $X_1 \times X_2$ is a Baire space. Let D and \tilde{D} be domains in $X_1 \times X_2$ of the following form:

$$D = \{(u, v) \in X_1 \times X_2: u \in U, v \in W_u\}$$

$$\tilde{D} = \{(u, v) \in X_1 \times X_2: u \in U, v \in V_u\},$$

where $U \subset X_1$, $W_u \subset V_u \subset X_2$ and every connected component of V_u contains a non-empty component of W_u . Suppose that $f: \tilde{D} \rightarrow Y$ is analytic in D and for every $u \in U$ $f_u: V_u \ni v \rightarrow f(u, v) \in Y$ is analytic in V_u . Then f is analytic in \tilde{D} .

Proof. Here we use the same methods as in the proof of the Generalized Hartogs Theorem for \mathbb{C}^n ([5], p. 141).

Fix $(u_0, v_0) \in \tilde{D}$. We shall prove that f is analytic in a neighbourhood of this point. By the definition of \tilde{D} we have: $u_0 \in U$, $v_0 \in V_{u_0}$. Fix an arbitrary point $w_0 \in W_{u_0}$ such that w_0 and v_0 are in the same component of V_{u_0} and take a polygonal line contained in V_{u_0} , joining w_0 to v_0 . We may assume, without loss of generality, that this polygonal line consists of one interval I . Now match W^* — a balanced neighbourhood of $0 \in X_2$ and V^* — a neighbourhood of u_0 in X_1 such that

- 1) $U^* \times (w_0 + W^*) \subset D$
- 2) $U^* \times (v + W^*) \subset \tilde{D}$, $v \in I$.

We choose these neighbourhoods in the following way. First we take U_1^* and W_1^* satisfying 1). Then to any point $v \in I$ we find a neighbourhood U_v^* of u_0 and a balanced neighbourhood W_v^* of $0 \in X_2$ such that $U_v^* \times (v + W_v^* + W_v^*) \subset \tilde{D}$. Since I is compact and $I \subset \bigcup_{v \in I} (v + W_v^*)$ we can find v_1, \dots, v_k such that

$I \subset \bigcup_{i=1}^k (v_i + W_{v_i}^*)$. Now it is enough to take $U^* = U_1^* \cap \bigcap_{i=1}^k U_{v_i}^*$ and $W^* = W_1^* \cap \bigcap_{i=1}^k W_{v_i}^*$. By hypothesis the function f is analytic in $U^* \times (w_0 + W^*)$. Let

$\varrho = \sup\{r > 0: r(v_0 - w_0) \in W^*\}$. If $\varrho > 1$ then $v_0 \in w_0 + W^*$, so f is analytic in a neighbourhood of (u_0, v_0) . Suppose that $\varrho \leq 1$ and take $v_1 = w_0 + \frac{\varrho}{2}(v_0 - w_0)$.

Then $v_1 \in I \cap (w_0 + W^*)$. Hence f is analytic in $U^* \times (v_1 + W^1)$, where W^1 is a balanced neighbourhood of $0 \in X_2$ such that $v_1 + W^1 \subset w_0 + W^*$. Moreover, by 2), $U^* \times (v_1 + W^*) \subset \tilde{D}$, so for every $u \in U^*$ f_u is analytic in $v_1 + W^*$. By Lemma 3.2. we find that f is analytic in $U^* \times (v_1 + W^*)$. Next we take $v_2 = v_1 + \frac{\varrho}{2}(v_0 - w_0)$.

Then $v_2 \in v_1 + W^*$ and $v_2 + W^2 \subset v_1 + W^*$, where W^2 is a sufficiently small neighbourhood of $0 \in X_2$. Applying once more Lemma 3.2. we obtain the

analyticity of f in $U^* \times (v_2 + W^*)$. Repeating this procedure after a finite number of steps, we shall reach the point v_0 , i.e. we shall obtain a point $v_k = w_0 + \frac{k\rho}{2}(v_0 - w_0) \in I$ such that $v_0 \in v_k + W^*$ and f will be analytic in $U^* \times (v_k + W^*)$.

Hence we shall obtain the analyticity of f in a neighbourhood of (u_0, v_0) .

THEOREM 3.2. *Suppose that X is a Fréchet space (i.e. metrizable, complete and locally convex). Let G be a domain in X containing a balanced neighbourhood Ω of $0 \in X$. Assume that $G^x = \{\lambda \in \mathbb{C}: \lambda x \in G\}$ is connected for every $x \in \Omega$. Then the following statement is true: If $f: G \rightarrow Y$ is analytic in Ω and $f_x: G^x \ni \lambda \rightarrow f(\lambda x) \in Y$ is analytic for every $x \in \Omega$ then $f \in \mathcal{A}(G, Y)$.*

Proof. Let $\tilde{D} = \{(x, \lambda) \in X \times \mathbb{C}: x \in \Omega, \lambda \in G^x\}$. Observe that \tilde{D} is a domain. Indeed, \tilde{D} is connected due to its form and \tilde{D} is open because of the continuity of operations in X . Let $D = \Omega \times B(0, 1)$. Since Ω is balanced we have $D \subset \tilde{D}$. Consider a new mapping $g: \tilde{D} \ni (x, \lambda) \rightarrow f(\lambda x) \in Y$. By hypothesis g is analytic in D and $g_x: G^x \ni \lambda \rightarrow g(x, \lambda) \in Y$ is analytic for every $x \in \Omega$. So we can apply Theorem 3.1. to the function g and to the domains \tilde{D} and D . Hence we come to the conclusion that $g \in \mathcal{A}(\tilde{D}, Y)$.

Fix any point $a \in G$. Then $a = \lambda'a'$, where $a' \in \Omega, \lambda' > 0$. Since $(a', \lambda') \in \tilde{D}$ g is analytic in a neighbourhood of (a', λ') . Observe that in a neighbourhood of a we have $f(x) = g \circ T_1 \circ T_2(x)$, where

$$\begin{aligned} T_2: X \ni x &\rightarrow (x, \lambda') \in X \times \mathbb{C}, \\ T_1: X \times (\mathbb{C} \setminus \{0\}) \ni (\xi, \tau) &\rightarrow (\frac{\xi}{\tau}, \tau) \in X \times \mathbb{C}. \end{aligned}$$

Moreover, $T_1 \circ T_2(a) = (a', \lambda')$. Therefore, by the analyticity of g in a neighbourhood of (a', λ') , f is analytic in a neighbourhood of a . The proof is completed.

4. Representations of analytic function in its Mittag-Leffler star. In this part X will denote a Fréchet space over \mathbb{C} , Y — as before — will be locally convex and sequentially complete.

Consider a series of homogeneous polynomials

$$(4.1) \quad f(x) = \sum_{n=0}^{\infty} f_n(x), \quad (f_n \in \mathfrak{P}^n(X, Y))$$

that converges in a neighbourhood of $0 \in X$. Then f , defined by (4.1), is analytic in the domain of convergence of series (4.1), i.e. in a balanced neighbourhood of $0 \in X$. Denote this domain by Ω .

Let

$$\begin{aligned} G_f = \bigcup \{G \subset X: 1) G \text{ is a starlike (with respect to } 0) \text{ domain,} \\ 2) \exists \tilde{f} \in \mathcal{A}(G, Y): \tilde{f}|_{G \cap \Omega} = f|_{G \cap \Omega}\}. \end{aligned}$$

Definition 4.1. The domain G_f is called the *Mittag-Leffler star* of the function f .

G_f is the largest starlike domain such that f can be analytically continued to G_f . In particular the domain Ω as a balanced domain (and then starlike) is contained in G_f . Theorems concerning weak analytic continuation (e.g. th. 2, [12]) imply the equality $G_f = \text{int} \bigcap_{u \in Y'} G_{u \circ f}$.

Theorems 4.1, 4.2 and 4.4 which will be stated below give the analytic continuation of series (4.1) to the Mittag-Leffler star G_f .

THEOREM 4.1. *Let*

$$(i) \{\varphi_k\}_{k \in \mathcal{N}} \subset \mathcal{O}_{\mathbb{C}}, \varphi_k(\lambda) = \sum_{\nu=0}^{\infty} a_{k\nu} \lambda^{\nu}, \lambda \in \mathbb{C}, k \in \mathcal{N},$$

$$(ii) \varphi_k(\lambda) \xrightarrow[k \rightarrow \infty]{} (1-\lambda)^{-1} \text{ uniformly on compact subsets of } \mathbb{C} \setminus [1, \infty).$$

(iii) $f \subset X \times Y$ is an analytic mapping in a neighbourhood of $0 \in X$, given by series (4.1).

Put

$$\Phi_k(x) = \sum_{\nu=0}^{\infty} a_{k\nu} f_{\nu}(x), \quad x \in X.$$

Then

$$\Phi_k(x) \xrightarrow[k \rightarrow \infty]{} f(x), \quad x \in G_f$$

and for every $q \in \mathcal{S}(Y)$

$$q \circ (\Phi_k - f) \xrightarrow[k \rightarrow \infty]{} 0$$

locally uniformly in G_f .

Proof. Take an arbitrary point $x_0 \in G_f \setminus \{0\}$ and a starlike (with respect to 0) positively oriented regular Jordan curve $\gamma \subset G^{x_0} = \{\lambda \in \mathbb{C}: \lambda x_0 \in G_f\}$ so that the points $0, 1 \in \text{int} \gamma$ (int γ denotes the bounded component of $\mathbb{C} \setminus \gamma$). Repeating the argument of the proof of Theorem 1, [6] we obtain

$$f(x) = (2\pi i)^{-1} \int_{\gamma} \lambda^{-1} (1-\lambda^{-1})^{-1} f(\lambda x) d\lambda, \quad x \in U$$

where U is a neighbourhood of $0 \in X$ so that

$$\Delta = \{\lambda x: \lambda \in \overline{\text{int} \gamma}, x \in U\} \subset G_f.$$

Since $\{\lambda^{-1}: \lambda \in \gamma\}$ is a compact set contained in $\mathbb{C} \setminus \{0\}$ then

$$\varphi_k(\lambda^{-1}) \xrightarrow[k \rightarrow \infty]{} (1-\lambda^{-1})^{-1}, \quad \lambda \in \gamma$$

the convergence being uniform on γ .

Hence

$$f(x) = (2\pi i)^{-1} \int_{\gamma} [\lim_{k \rightarrow \infty} \varphi_k(\lambda^{-1})] \lambda^{-1} f(\lambda x) d\lambda, \quad x \in U.$$

Observe that for every $x \in U$ the function $\{\lambda \rightarrow \lambda^{-1}f(\lambda x)\}$ is analytic in a neighbourhood of γ and so it is q -bounded on γ for any $q \in \mathfrak{G}(Y)$. Then for any $q \in \mathfrak{G}(Y)$ and for any $x \in U$

$$q[\varphi_k(\lambda^{-1})\lambda^{-1}f(\lambda x) - \lambda^{-1}(1 - \lambda^{-1})^{-1}f(\lambda x)] \xrightarrow[k \rightarrow \infty]{} 0$$

uniformly on γ . Hence

$$f(x) = \lim_{k \rightarrow \infty} (2\pi i)^{-1} \int_{\gamma} \varphi_k(\lambda^{-1}) \lambda^{-1} f(\lambda x) d\lambda, \quad x \in U.$$

Moreover for arbitrary $x \in U$, $q \in \mathfrak{G}(Y)$ and $k \in \mathcal{N}$

$$q\left(\sum_{\nu=0}^{\infty} a_{k\nu} \lambda^{-\nu} \lambda^{-1} f(\lambda x)\right) \xrightarrow[p \rightarrow \infty]{} 0$$

uniformly on γ and so we have

$$\begin{aligned} f(x) &= \lim_{k \rightarrow \infty} \sum_{\nu=0}^{\infty} a_{k\nu} (2\pi i)^{-1} \int_{\gamma} \lambda^{-\nu-1} f(\lambda x) d\lambda \\ &= \lim_{k \rightarrow \infty} \sum_{\nu=0}^{\infty} a_{k\nu} f_{\nu}(x) = \lim_{k \rightarrow \infty} \Phi_k(x), \quad x \in U. \end{aligned}$$

To prove that the convergence is locally uniform fix an arbitrary $q \in \mathfrak{G}(Y)$ and take a neighbourhood $V \subset U$ of x_0 so that the mapping $\{(\lambda, x) \rightarrow |\lambda|^{-1}q(f(\lambda x))\}$ is bounded on $\gamma \times V$. It is possible to find such a neighbourhood since γ is compact and multiplication by scalars is continuous in X . For $x \in V$ we have

$$\begin{aligned} q(f(x) - \Phi_k(x)) &= q\left(\frac{1}{2\pi i} \int_{\gamma} (1 - \lambda^{-1})^{-1} \lambda^{-1} f(\lambda x) d\lambda - \frac{1}{2\pi i} \int_{\gamma} \varphi_k(\lambda^{-1}) \lambda^{-1} f(\lambda x) d\lambda\right) \\ &\leq \frac{1}{2\pi} M_q \int_{\gamma} |(1 - \lambda^{-1})^{-1} - \varphi_k(\lambda^{-1})| |d\lambda|, \end{aligned}$$

where M_q does not depend on $x \in V$. Since $|(1 - \lambda^{-1})^{-1} - \varphi_k(\lambda^{-1})| \rightarrow 0$ when $k \rightarrow \infty$, uniformly on γ , then $q(f(x) - \Phi_k(x)) \rightarrow 0$ when $k \rightarrow \infty$, uniformly on V . The proof is completed.

Remark 4.1. It follows from the Runge approximation theorem that there exists a sequence $\{\varphi_k\}_{k \in \mathcal{N}}$, consisting of polynomials, satisfying the hypothesis of Theorem 4.1. We obtain then the following theorem of a triangular array (for $X = \mathbb{C}^n$ th. 1, [6])

THEOREM 4.2. *There is a triangular array of complex numbers $c_0^k, c_1^k, \dots, c_{\nu_k}^k$ ($k = 0, 1, \dots$) such that*

$$f(x) = \sum_{k=0}^{\infty} [c_0^k f_0(x) + c_1^k f_1(x) + \dots + c_{\nu_k}^k f_{\nu_k}(x)], \quad x \in G_f$$

and for every $q \in \mathfrak{G}(Y)$

$$q\left(f(x) - \sum_{k=0}^p [c_0^k f_0(x) + \dots + c_{\nu_k}^k f_{\nu_k}(x)]\right) \xrightarrow{p \rightarrow \infty} 0$$

locally uniformly in G_f .

Theorems 4.1 and 4.2 give the analytic continuation of function f to G_f as a limit of a sequence of entire functions. We obtain another representation of analytic continuation of function f by applying Borel's method of summability of power series.

Given a function f represented by series (4.1) we write

$$(4.2) \quad F_k(x) = \sum_{\nu=0}^{\infty} \left[\Gamma\left(1 + \frac{\nu}{k}\right) \right]^{-1} f_{\nu}(x), \quad x \in X,$$

and call the k -th function associated with f ($k \in \mathcal{N}$).

LEMMA 4.1. Let $q \in \mathfrak{G}(Y)$ and $k \in \mathcal{N}$. For every $x_0 \in X$ there exist a neighbourhood V of x_0 and constants M_k^q, a_k^q such that

$$q(F_k(\lambda x)) \leq M_k^q \exp(a_k^q |\lambda|^k), \quad x \in V, \lambda \in \mathbb{C}.$$

Proof. Since X is a Baire space one can prove ([3], Prop. 5.2) that the series (4.1) converges normally at zero, i.e. for any $q \in \mathfrak{G}(Y)$ there exists a neighbourhood W of $0 \in X$ such that $\sum_{\nu=0}^{\infty} \|q \circ f_{\nu}\|_W < \infty$ ($\|q \circ f_{\nu}\|_W = \sup \{q \circ f_{\nu}(x), x \in W\}$). Take such W and fix any $\theta > 1$. Then $\|q \circ f_{\nu}\|_W \leq m_q \theta^{\nu}$ with sufficient m_q . For any $x_0 \in X$ there exist $t > 0$ and a neighbourhood V of x_0 such that $tV \subset W$. Therefore $q(f_{\nu}(x)) \leq m_q (\theta/t)^{\nu}$ when $x \in V$. So

$$\begin{aligned} q(F_k(\lambda x)) &\leq \sum_{\nu=0}^{\infty} q(f_{\nu}(x)) |\lambda|^{\nu} / \Gamma\left(1 + \frac{\nu}{k}\right) \\ &\leq m_q \sum_{\nu=0}^{\infty} \left(\frac{\theta}{t}\right)^{\nu} |\lambda|^{\nu} / \Gamma\left(1 + \frac{\nu}{k}\right), \quad x \in V, \quad \lambda \in \mathbb{C}. \end{aligned}$$

But

$$\sum_{\nu=0}^{\infty} \left(\frac{\theta}{t}\right)^{\nu} |\lambda|^{\nu} / \Gamma\left(1 + \frac{\nu}{k}\right) = E_k\left(\frac{\theta}{t} |\lambda|\right),$$

where E_k is a function of Mittag-Leffler type, in particular E_k is a function of order k and type 1 (see [7]). Hence

$$E_k(\theta t^{-1} |\lambda|) \leq M_k \exp(\theta^{k+1} t^{-k} |\lambda|^k), \quad \lambda \in \mathbb{C}, x \in V,$$

with suitable constant M_k . Now it is sufficient to put $M_k^q = m_q M_k$ and $a_k^q = \theta^{k+1} t^{-k}$.

Given $q \in \mathfrak{G}(Y)$ and $k \in \mathcal{N}$ write

$$H_k^q(x_0) = \overline{\lim_{x \rightarrow x_0}} \overline{\lim_{t \rightarrow \infty}} t^{-k} \ln q(F_k(tx)), \quad x_0 \in X.$$

and call H_k^q the *regularized radial q -indicator* of the function F_k .

By Lemma 4.1 the family of plurisubharmonic functions

$$\{\{x \rightarrow t^{-k} \ln q(F_k(tx))\}, t \geq t_0 > 0\}$$

is locally uniformly bounded from above in X . Therefore H_k^q is plurisubharmonic in X (see [10], th. 2.2.3). It follows from its definition that the function H_k^q is positively homogeneous of order k , i.e. $H_k^q(tx) = t^k H_k^q(x)$, $t > 0$.

Now we shall recall some definitions which are contained in [6].

Given $\theta \in (-\pi, \pi]$, $k \in \mathcal{N}$ and $c \geq 0$ consider the curve

$$L_k(\theta, c) = \left\{ \zeta \in \mathbb{C}: \operatorname{Re}(\zeta e^{-i\theta})^k = c, |\operatorname{Arg}(\zeta e^{-i\theta})| \leq \frac{\pi}{2k} \right\}$$

If $k = 1$ we admit also $c < 0$.

The curve $L_k(\theta, c)$ cuts the complex plane in two disjoint domains: $D_k^*(\theta, c)$ and $D_k(\theta, c)$ containing the intervals

$$\{0 < |\zeta| < c^{1/k}, \operatorname{Arg} \zeta = \theta\} \quad \text{and} \quad \{c^{1/k} < |\zeta| < \infty, \operatorname{Arg} \zeta = \theta\}$$

respectively.

The closed set $\overline{D_k^*(\theta, c)}$ is called an *elementary k -convex set*. We say that a closed set $M \subset \mathbb{C}$ is *k -convex* if it is the intersection of a family of elementary k -convex sets. By a *k -convex hull* of a set $M \subset \mathbb{C}$ we mean the intersection of all elementary k -convex sets containing M .

Let $W \subset \mathbb{C}$ be a bounded k -convex set (hence W is compact). Write $W(k, \theta) = W \cap D_k(\theta, 0)$. We define the function κ_k as follows:

$$\begin{aligned} \kappa_1(\theta) &= \max \{ \operatorname{Re}(\zeta e^{-i\theta}) : \zeta \in W \} \\ \kappa_k(\theta) &= \begin{cases} \max_{\zeta \in W(k, \theta)} \operatorname{Re}(\zeta e^{-i\theta})^k & \text{when } W(k, \theta) \neq \emptyset \\ 0 & \text{when } W(k, \theta) = \emptyset, \end{cases} \quad \text{for } k > 1 \end{aligned}$$

and call κ_k a *k -supporting function* of the set W .

Assume that the sum of series

$$\Phi(\lambda) = \sum_{\nu=0}^{\infty} b_\nu \lambda^\nu / \Gamma\left(1 + \frac{\nu}{k}\right), \quad \lambda \in \mathbb{C}$$

is an entire function of order k and type σ ($k, \sigma \in (0, \infty)$). Write

$$\varphi_k(\lambda) = \sum_{\nu=0}^{\infty} b_\nu \lambda^{\nu-1}, \quad |\lambda| > \sigma^{1/k}$$

The function φ_k is called \mathfrak{B}_k transformation of Φ (or the generalized Borel transformation of Φ).

LEMMA 4.2. *Let*

$$\psi(\lambda) = \sum_{\nu=0}^{\infty} \lambda^{-\nu-1} a_{\nu} \quad (a_{\nu} \in Y)$$

be a function of one complex variable analytic in a neighbourhood of infinity $\{|\lambda| > r\}$. Put

$$\Psi(\lambda) = \sum_{\nu=0}^{\infty} \frac{\lambda^{\nu}}{\Gamma\left(1 + \frac{\nu}{k}\right)} a_{\nu}, \quad \lambda \in \mathbb{C}.$$

Let W_k denote the smallest starlike set so that ψ can be analytically continued to $\mathbb{C} \setminus W_k$. Given $q \in \mathfrak{S}(Y)$ write

$$h_k^q(\theta) = \overline{\lim}_{t \rightarrow \infty} t^{-k} \ln q(\Psi(te^{i\theta})).$$

If \tilde{W}_k is the k -convex hull of W_k and $\tilde{\kappa}_k$ is the k -supporting function of \tilde{W}_k then for every $q \in \mathfrak{S}(Y)$

$$h_k^q(\theta) \leq \tilde{\kappa}_k(-\theta), \quad \theta \in (-\pi, \pi)$$

Remark 4.2. $W_k = \overline{\bigcup_{u \in Y'} W_{u,k}}$ where $W_{u,k}$ denotes the smallest starlike set such that $u \circ \psi$ is holomorphic in $\mathbb{C} \setminus W_{u,k}$ (see [12], th. 2).

Proof of Lemma 4.2. For any $u \in Y'$ $u \circ \psi$ is analytic in $\mathbb{C} \setminus W_k$. Note that $u \circ \psi$ is the \mathfrak{B}_k -transformation of $u \circ \Psi$. Since $u \circ \psi$ is analytic in the domain $\{|\lambda| > r\}$ the order ρ_u of $u \circ \Psi$ does not exceed k and if $\rho_u = k$ the type of $u \circ \Psi$ is finite ([7], Lemma 6.1). So we can apply the generalized Polya Theorem (see [7], th. 6.6 or [6], th. 5) to the function $\Phi = u \circ \Psi$. Since $W_{u,k} \subset W_k$ for any $u \in Y'$ we have

$$\overline{\lim}_{t \rightarrow \infty} t^{-k} \ln |u \circ \Psi(te^{i\theta})| \leq \tilde{\kappa}_k(-\theta), \quad u \in Y'.$$

Hence for every $\delta > 0$

$$t^{-k} \ln |u \circ \Psi(te^{i\theta})| \leq \tilde{\kappa}_k(-\theta) + \delta, \quad t \geq t_0, u \in Y'.$$

So

$$|u(\Psi(te^{i\theta})e^{-(\tilde{\kappa}_k(-\theta)+\delta)t^k})| \leq 1, \quad t \geq t_0, u \in Y'$$

and then

$$q(\Psi(te^{i\theta})e^{-(\tilde{\kappa}_k(-\theta)+\delta)t^k}) \leq M_q, \quad t \geq t_0, q \in \mathfrak{S}(Y).$$

Therefore

$$t^{-k} \ln q(\Psi(te^{i\theta})) \leq t^{-k} \ln M_q + \tilde{\kappa}_k(-\theta) + \delta, \quad q \in \mathfrak{S}(Y).$$

