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# Generating path factors of a tree

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**1. Introduction.** Some recent papers concern the problem of covering the vertices in a graph, say $G$, by disjoint paths. These covering paths form a path factor of $G$. The minimal number of these paths, say $\pi_0(G)$, is called in [2] and [8, 9] the path-to-point covering number of $G$ and the vertex-path partition number of $G$, respectively. Another number, say $s_H(G)$, which is equal to 0 for a Hamiltonian graph $G$ and to $\pi_0(G)$ for a non-Hamiltonian graph $G$ with $n$ ($\geq 2$) vertices is called in [4] and [8, 9] the Hamiltonian completion number of $G$ and Hamiltonian shortage in $G$, respectively. Sufficient conditions for the inequality $\pi_0(G) \leqslant s$ (or $s_H(G) \leqslant s$), $s \in N$, are given in [6, 7, 2, 8, 5, 3, 9], the most extensive lists of conditions being given in [7] and [9]. The corresponding conditions of [3] were published earlier in [7] in 1974. It is obvious (cf. [2, 4]) that, for a connected graph $G$,

$$\pi_0(G) = \min\{\pi_0(T)\colon\ T \text{ is a spanning tree of } G\}.$$

Therefore determining $\pi_0$ for a tree is of special importance. Algorithms for evaluating $\pi_0$ for a tree $T$ are given in [2, 4, 8]. The problem of finding all maximal path factors of a graph is posed in [8]. In this paper we show how the generating path factors of a tree, which has a vertex of degree 2, may be simplified. A labelling procedure, presented in [8], is modified in [10] and [11] so as to be helpful in determining all maximal path factors of a tree.

**2. Definitions.** The symbol $:=$ denotes an equality on the strength of a definition. Given a set $Z$, the symbol $|Z|$ denotes its cardinality. A *graph* $G$ is the pair of two sets $V$ and $E$, in symbols $G = \langle V, E\rangle$, $E$ being a certain subcollection of the collection $P_2(V)$ of two element subsets of $V$ with $V \cap P_2(V) = \emptyset$. $V$ and $E$ are the *vertex set* $V(G)$ and the *edge set* $E(G)$ of the graph $G$, respectively. The numbers $|V(G)|$ and $|E(G)|$ of vertices and edges of $G$ are the *order* and the *size* of the graph $G$, respectively. Two elements $u_1, u_2 \in V \cup E$ are called incident in $G$ if $u_i \in u_j \in E$ and $\{i, j\} = \{1, 2\}$ (then $u_i$ is a vertex and $u_j$ is an edge of $G$). The number of edges incident in $G$ to a vertex $x$ is the *degree* of $x$.

A graph $G_1 = \langle V_1, E_1\rangle$ is called a *subgraph* of the graph $G$, in symbols $G_1 \subseteq G$, if $V_1 \subseteq V$ and $E_1 \subseteq E$. Given two graphs $G_i = \langle V_i, E_i\rangle$, $i = 1, 2$, being subgraphs of certain graph, their *union* and *intersection* are defined as follows:

$$G_1 \cup G_2 = \langle V_1 \cup V_2,\ E_1 \cup E_2 \rangle,$$

$$G_1 \cap G_2 = \langle V_1 \cap V_2,\ E_1 \cap E_2 \rangle.$$

A *factor* is a subgraph with the same vertex set. A *path*, $P_k$, is a graph with $k$ vertices $x_1, x_2, \ldots, x_k$ and the edge set

$$E(P_k) = \{\{x_i, x_{i+1}\}:\ i = 1, 2, \ldots, k-1\}, \qquad k \geqslant 1\ (E(P_1) = \emptyset),$$

$P_1$ being a *trivial path*. The vertices $x_1$ and $x_k$ are *end-vertices* of $P_k$, and $P_k$ is said to *connect* $x_1$ and $x_k$. A graph is *connected* if any two of its vertices are connected by a path being a subgraph of the graph. A *component* of the graph $G$ is any maximal connected subgraph of $G$, $k(G)$ denotes the number of components of $G$.

A *tree* $T$ is a connected graph with $|E(T)| = |V(T)|-1$ (so a tree is a connected graph without circuits).

By a *path factor* of a tree $T$ we mean a factor $W$ of $T$, each component of which is a path (possibly trivial). Path factors are called in [2] island decompositions.

Let $\mathcal{W}(T)$ be the collection of all path factors of $T$. Consider the number

$$\pi_0(T) := \min\{k(W):\ W \in \mathcal{W}(T)\}.$$

A path factor $W$ of $T$ is *maximal* if $k(W) = \pi_0(T)$. A maximal path factor of $T$ was called in [8] an optimal path system in $T$.

In what follows $\mathcal{S}(T)$ and $S$ will denote the collection of all maximal path factors of $T$ and an element of this collection, respectively. Observe that

$$S \in \mathcal{S}(T) \Leftrightarrow S \in \mathcal{W}(T) \quad \text{and} \quad |E(S)| = \max\{|E(W)|:\ W \in \mathcal{W}(T)\}.$$

Hence, each maximal path factor is of maximal size among all path factors of $T$. This follows from the following obvious formula

$$k(W) = |V(W)| - |E(W)| \quad \text{for } W \in \mathcal{W}(T),$$

and from the definition of a maximal path factor.

**3. Decomposing the tree with respect to a vertex of degree 2.** A pair of subtrees $T_1$ and $T_2$ of a tree $T$ such that

$$T = T_1 \cup T_2, \qquad V(T_1) \cap V(T_2) = \{v\}$$

with $v$ being a vertex of degree 2 in $T$ is the *decomposition* of the tree $T$ with respect to $v$.

THEOREM 1. *If $T_1$, $T_2$ is a decomposition of a tree $T$ with respect to a vertex of degree 2 in $T$ then the collection $\mathcal{S}(T)$ of all maximal path factors of $T$ induces* *the collection of maximal path factors of each* $T_i$, $i = 1, 2$; *namely*

$$\mathcal{S}(T_i) = \{S \cap T_i\colon\ S \in \mathcal{S}(T)\}\,, \qquad i = 1, 2\,.$$

The proof of Theorem 1 depends on the following

LEMMA 1. *Under the assumptions of Theorem* 1 *if* $S \in \mathcal{S}(T)$ *and* $S_i \in \mathcal{S}(T_i)$ *then, for* $\{i, j\} = \{1, 2\}$, *the union of* $S_i$ *and* $S \cap T_j$ *is a maximal path factor of* $T$, i.e.,

$$S_i \cup (S \cap T_j) \in \mathcal{S}(T) \qquad \text{for } \{i, j\} = \{1, 2\}\,.$$

Proof. Let $\{i, j\} = \{1, 2\}$ and let

$$\bar{S} = S_i \cup (S \cap T_j)\,.$$

Components of $\bar{S}$ are clearly paths and $V(\bar{S}) = V(T)$. Therefore $\bar{S} \in \mathcal{W}(T)$. Suppose that $\bar{S} \notin \mathcal{S}(T)$, that is, $k(\bar{S}) > \pi_0(T)$. Hence, since $k(S) = \pi_0(T)$, $S \cap T_j = \bar{S} \cap T_j$, and $S \cap T_i \in \mathcal{W}(T_i)$ therefore $k(\bar{S} \cap T_i) > k(S \cap T_i) \geqslant \pi_0(T_i)$, which s impossible since $\bar{S} \cap T_i = S_i \in \mathcal{S}(T_i)$.

Proof of Theorem 1. It suffices to consider the case $i = 1$. Let

$$\tilde{\mathcal{S}}(T_1) = \{S \cap T_1\colon\ S \in \mathcal{S}(T)\}\,.$$

Hence for any $\tilde{S}_1 \in \tilde{\mathcal{S}}(T_1)$ there is $S \in \mathcal{S}(T)$ such that $\tilde{S}_1 = S \cap T_1$. We shall show that $\tilde{S}_1 \in \mathcal{S}(T_1)$. To do this, choose some $S_1 \in \mathcal{S}(T_1)$. Then, by Lemma 1,

$$S^* := S_1 \cup (S \cap T_2)$$

is a maximal path factor of $T$. Since $S^* \cap T_2 = S \cap T_2$ therefore

$$k(S^* \cap T_1) = k(S \cap T_1)\,. \quad \text{So } k(S_1) = k(\tilde{S}_1) \text{ whence } \tilde{S}_1 \in \mathcal{S}(T_1)$$

and $\tilde{\mathcal{S}}(T_1) \subseteq \mathcal{S}(T_1)$.

To prove the converse inclusion, choose any $S_1 \in \mathcal{S}(T_1)$ and any $S \in \mathcal{S}(T)$. By Lemma 1,

$$S' := S_1 \cup (S \cap T_2)$$

is a maximal path factor of $T$ and

$$S' \cap T_1 = S_1\,.$$

Therefore $S_1 \in \tilde{\mathcal{S}}(T_1)$, Q.E.D.

THEOREM 2. *If* $T_1, T_2$ *is a decomposition of a tree* $T$ *with respect to a vertex* $v$ *of degree* 2 *in* $T$, *then the collection* $\mathcal{S}(T)$ *of all maximal path factors in* $T$ *coincides with the collection*

$$\mathcal{S}_{12} := \{S_1 \cup S_2\colon\ S_i \in \mathcal{S}(T_i)\,, \qquad i = 1, 2\}\,,$$

$\mathcal{S}(T_i)$ *being the collection of maximal path factors of* $T_i$, $i = 1, 2$.

Proof. Let $S_1 \in \mathcal{S}(T_1)$ and $S_2 \in \mathcal{S}(T_2)$. Choose an $S' \in \mathcal{S}(T)$. By Lemma 1, $S_1 \cup (S' \cap T_2) \in \mathcal{S}(T)$. Hence, similarly

$$S_2 \cup S_1 = S_2 \cup [(S_1 \cup (S' \cap T_2)) \cap T_1] \in \mathcal{S}(T) .$$

Therefore $\mathcal{S}_{12} \subseteq \mathcal{S}(T)$.

Now let $S \in \mathcal{S}(T)$. Hence, by Theorem 1,

$$S \cap T_i \in \mathcal{S}(T_i) , \quad i = 1, 2 \quad \text{and } S = (S \cap T_1) \cup (S \cap T_2) .$$

Therefore $S \in \mathcal{S}_{12}$ and $\mathcal{S}(T) \subseteq \mathcal{S}_{12}$, Q.E.D.

Remark 1. If $T_1, T_2$ is a decomposition of $T$ with respect to a vertex of degree 2 in $T$ then

$$\mathcal{W}(T_i) = \{W \cap T_i\colon\ W \in \mathcal{W}(T)\} ,$$

and

$$\mathcal{W}(T) = \{W_1 \cup W_2\colon\ W_i \in \mathcal{W}(T_i) , \quad i = 1, 2\}$$

(cf. the above theorems). Moreover, by Theorem 2, there are the following quantitative relations:

$$|\mathcal{S}(T)| = |\mathcal{S}(T_1)| \cdot |\mathcal{S}(T_2)| ,$$

and

$$\pi_0(T) = \pi_0(T_1) + \pi_0(T_2) - 1 .$$

Remark 2. Both these theorems concern the decomposition of a tree $T$ with respect to any vertex of degree 2. It is clear, however, that simplification is essential in the case of such a vertex of degree 2, which does not lie on a hanging string. (A *string* in $T$ is a non-trivial path whose each inner vertex (if any) is of degree 2 in $T$ and both end-vertices are of degrees different from 2 in $T$. A string is *hanging* if one of its end-vertices is of degree 1 in $T$). So if the tree $T$ has a non-hanging string with an inner vertex of degree 2, then it is advisable to choose one vertex of degree 2 from each such string and to decompose $T$ with respect to chosen vertices into a number of subtrees, say $T_1, T_2, \ldots, T_k$.

Then

$$|\mathcal{S}(T)| = \prod_{i=1}^{k} |\mathcal{S}(T_i)| ,$$

$$\pi_0(T) = \sum_{i=1}^{k} \pi_0(T_i) - k + 1 ,$$

and

$$\mathcal{S}(T) = \{\bigcup_{i=1}^{k} S_i\colon\ S_i \in \mathcal{S}(T_i),\ i = 1, 2, \ldots, k\} .$$

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