

## Uniqueness of solutions of the Cauchy problem for first order differential-functional equations

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**Abstract.** We consider a system of first order differential-functional equations of the form

$$(1) \quad u_{z_j}^i(X, Y) = f_j^i(X, Y, U(X, Y), u_Y^i(X, Y), U(X, \cdot))$$

$$(i = 1, \dots, m; j = 1, \dots, p)$$

with initial data

$$(2) \quad u^i(\overset{\circ}{X}, Y) = \varphi^i(Y) \quad (i = 1, \dots, m)$$

where  $X = (x_1, \dots, x_p)$ ,  $Y = (y_1, \dots, y_n)$ ,  $U = (u^1, \dots, u^m)$  and  $u_Y^i = (u_{y_1}^i, \dots, u_{y_n}^i)$ . The function  $U(X, Y)$  is defined in a pyramid  $P$  and for a fixed  $X$  we denote by  $U(X, \cdot)$  the function  $Y \rightarrow U(X, Y)$  as an element of the space of continuous functions from the set  $\{Y: (X, Y) \in P\}$  to  $R^m$ . For a solution of problem (1), (2) of class  $\mathcal{D}$  in the pyramid  $P$  the following questions are dealt with: estimate of a solution, estimate of the difference between two solutions, the uniqueness of a solution and its continuous dependence on the initial data and on the right-hand sides of the system. Theorems to be proved are known (see [1]) if the right-hand sides of (1) do not depend functionally on  $U(X, \cdot)$ . Similar theorems for parabolic differential-functional equations are proved in the paper [2].

**1. Definitions.** In the space  $(x_1, \dots, x_p, y_1, \dots, y_n)$  we denote by  $P$  the pyramid

$$\sum_{r=1}^p |x_r - \overset{\circ}{x}_r| < \gamma, |y_k - \overset{\circ}{y}_k| \leq a_k - L \sum_{r=1}^p |x_r - \overset{\circ}{x}_r| \quad (k = 1, \dots, n)$$

where  $0 \leq L < +\infty$ ,  $0 < a_k < +\infty$ ,  $\gamma \leq \min_k (a_k/L)$ . The point  $X = (x_1, \dots, x_p)$ , such that  $\sum_{r=1}^p |x_r - \overset{\circ}{x}_r| < \gamma$ , being fixed we put  $P_X = \{Y: (X, Y) \in P\}$ .

Let  $C_m(P_X)$  stand for the space of continuous functions  $Z(Y) = (z^1(Y), \dots, z^m(Y))$  from  $P_X$  to  $R^m$  with the norm

$$\|Z\| = \max_i \max \{|z^i(Y)|; Y \in P_X\}$$

A function  $u(X, Y)$  will be called the function of class  $\mathcal{D}$  in the pyramid  $P$  if  $u(X, Y)$  is continuous in  $P$ , possesses Stolz's differential with regard to  $(X, Y)$  on its side surface and has first derivatives with respect to  $Y$  and Stolz's differential with respect to  $X$  in its interior.

If, moreover, the derivatives  $u_{x_i}(X, Y)$  ( $i = 1, \dots, p$ ) are continuous with respect to  $(X, Y)$  for  $X = X_0$ , then  $u(X, Y)$  will be called the function of class  $\mathcal{D}_0$ .

Assumptions  $H_1$ . The function  $\sigma(t, y)$  will be said to satisfy Assumptions  $H_1$  if it is non-negative and continuous in the domain  $t \geq 0, y \geq 0$ .

For  $\eta \geq 0$  we denote by  $\omega(t, \eta)$  the right-hand maximum solution (see [1] § 5) through  $(0, \eta)$  of the ordinary equation

$$(1.1) \quad \frac{dy}{dt} = \sigma(t, y)$$

Assumptions  $H_2$ . The function  $\sigma(t, y)$  is said to satisfy Assumptions  $H_2$  if in the domain  $t > 0, y \geq 0$  it is non-negative and continuous,  $\sigma(t, 0) = 0$  and  $y(t) = 0$  is the unique solution of (1.1) satisfying the condition

$$\lim_{t \rightarrow 0} y(t) = 0$$

Assumptions  $H_3$ . The function  $\sigma(t, y)$  is said to satisfy Assumptions  $H_3$  if in the domain  $t > 0, y \geq 0$  it is non-negative and continuous,  $\sigma(t, 0) = 0$  and  $y(t) = 0$  is the unique solution of (1.1) satisfying the conditions

$$\lim_{t \rightarrow 0} y(t) = \lim_{t \rightarrow 0} \frac{y(t)}{t} = 0$$

**2. Comparison theorems for a system of differential-functional inequalities.** In this chapter we will prove three comparison theorems for a system of differential-functional inequalities in the case  $p = 1$ .  $x_1$  is now simply denoted by  $x$ .

**THEOREM 2.1.** *Let the functions  $U(x, Y) = (u^1(x, Y), \dots, u^m(x, Y))$  be of class  $\mathcal{D}$  in the pyramid*

$$(2.1) \quad |x - x_0| < \gamma, \quad |y_k - \overset{\circ}{y}_k| \leq a_k - L|x - x_0| \quad (k = 1, \dots, n)$$

where  $0 \leq L < +\infty, 0 < a_k < +\infty, \gamma < \min_k (a_k/L)$ . Suppose the initial inequalities

$$(2.2) \quad |u^i(x_0, Y)| \leq \eta \quad (i = 1, \dots, m)$$

and the differential-functional inequalities

$$(2.3) \quad |u_x^i| \leq \sigma(|x - x_0|, \|U(x, \cdot)\|) + L \sum_{k=1}^n |u_{y_k}^i| \quad (i = 1, \dots, m)$$

