

Uniqueness of solutions of the Cauchy problem for first order differential-functional equations

by M. NOWOTARSKA

Abstract. We consider a system of first order differential-functional equations of the form

$$(1) \quad u_{z_j}^i(X, Y) = f_j^i(X, Y, U(X, Y), u_Y^i(X, Y), U(X, \cdot))$$

$$(i = 1, \dots, m; j = 1, \dots, p)$$

with initial data

$$(2) \quad u^i(\overset{\circ}{X}, Y) = \varphi^i(Y) \quad (i = 1, \dots, m)$$

where $X = (x_1, \dots, x_p)$, $Y = (y_1, \dots, y_n)$, $U = (u^1, \dots, u^m)$ and $u_Y^i = (u_{y_1}^i, \dots, u_{y_n}^i)$. The function $U(X, Y)$ is defined in a pyramid P and for a fixed X we denote by $U(X, \cdot)$ the function $Y \rightarrow U(X, Y)$ as an element of the space of continuous functions from the set $\{Y: (X, Y) \in P\}$ to R^m . For a solution of problem (1), (2) of class \mathcal{D} in the pyramid P the following questions are dealt with: estimate of a solution, estimate of the difference between two solutions, the uniqueness of a solution and its continuous dependence on the initial data and on the right-hand sides of the system. Theorems to be proved are known (see [1]) if the right-hand sides of (1) do not depend functionally on $U(X, \cdot)$. Similar theorems for parabolic differential-functional equations are proved in the paper [2].

1. Definitions. In the space $(x_1, \dots, x_p, y_1, \dots, y_n)$ we denote by P the pyramid

$$\sum_{r=1}^p |x_r - \overset{\circ}{x}_r| < \gamma, |y_k - \overset{\circ}{y}_k| \leq a_k - L \sum_{r=1}^p |x_r - \overset{\circ}{x}_r| \quad (k = 1, \dots, n)$$

where $0 \leq L < +\infty$, $0 < a_k < +\infty$, $\gamma \leq \min_k (a_k/L)$. The point $X = (x_1, \dots, x_p)$, such that $\sum_{r=1}^p |x_r - \overset{\circ}{x}_r| < \gamma$, being fixed we put $P_X = \{Y: (X, Y) \in P\}$.

Let $C_m(P_X)$ stand for the space of continuous functions $Z(Y) = (z^1(Y), \dots, z^m(Y))$ from P_X to R^m with the norm

$$\|Z\| = \max_i \max \{|z^i(Y)|; Y \in P_X\}$$

A function $u(X, Y)$ will be called the function of class \mathcal{D} in the pyramid P if $u(X, Y)$ is continuous in P , possesses Stolz's differential with regard to (X, Y) on its side surface and has first derivatives with respect to Y and Stolz's differential with respect to X in its interior.

If, moreover, the derivatives $u_{x_i}(X, Y)$ ($i = 1, \dots, p$) are continuous with respect to (X, Y) for $X = X_0$, then $u(X, Y)$ will be called the function of class \mathcal{D}_0 .

Assumptions H_1 . The function $\sigma(t, y)$ will be said to satisfy Assumptions H_1 if it is non-negative and continuous in the domain $t \geq 0, y \geq 0$.

For $\eta \geq 0$ we denote by $\omega(t, \eta)$ the right-hand maximum solution (see [1] § 5) through $(0, \eta)$ of the ordinary equation

$$(1.1) \quad \frac{dy}{dt} = \sigma(t, y)$$

Assumptions H_2 . The function $\sigma(t, y)$ is said to satisfy Assumptions H_2 if in the domain $t > 0, y \geq 0$ it is non-negative and continuous, $\sigma(t, 0) = 0$ and $y(t) = 0$ is the unique solution of (1.1) satisfying the condition

$$\lim_{t \rightarrow 0} y(t) = 0$$

Assumptions H_3 . The function $\sigma(t, y)$ is said to satisfy Assumptions H_3 if in the domain $t > 0, y \geq 0$ it is non-negative and continuous, $\sigma(t, 0) = 0$ and $y(t) = 0$ is the unique solution of (1.1) satisfying the conditions

$$\lim_{t \rightarrow 0} y(t) = \lim_{t \rightarrow 0} \frac{y(t)}{t} = 0$$

2. Comparison theorems for a system of differential-functional inequalities. In this chapter we will prove three comparison theorems for a system of differential-functional inequalities in the case $p = 1$. x_1 is now simply denoted by x .

THEOREM 2.1. *Let the functions $U(x, Y) = (u^1(x, Y), \dots, u^m(x, Y))$ be of class \mathcal{D} in the pyramid*

$$(2.1) \quad |x - x_0| < \gamma, \quad |y_k - \overset{\circ}{y}_k| \leq a_k - L|x - x_0| \quad (k = 1, \dots, n)$$

where $0 \leq L < +\infty, 0 < a_k < +\infty, \gamma < \min_k (a_k/L)$. Suppose the initial inequalities

$$(2.2) \quad |u^i(x_0, Y)| \leq \eta \quad (i = 1, \dots, m)$$

and the differential-functional inequalities

$$(2.3) \quad |u_x^i| \leq \sigma(|x - x_0|, \|U(x, \cdot)\|) + L \sum_{k=1}^n |u_{y_k}^i| \quad (i = 1, \dots, m)$$

are satisfied in the pyramid (2.1), where $\sigma(t, y)$ satisfies Assumptions H_1 . Let $\omega(t, \eta)$ be the right-hand maximum solution of (1.1) through $(0, \eta)$ and assume it to be defined in the interval $[0, \alpha_0)$.

Under these assumptions inequalities

$$(2.4) \quad |u^i(x, Y)| \leq \omega(|x - x_0|, \eta) \quad (i = 1, \dots, m)$$

hold true in the pyramid (2.1) for $|x - x_0| < \min(\gamma, \alpha_0)$.

Proof. Since the assumptions of the theorem are invariant under the mapping $\xi = -x + 2x_0$, it is sufficient to prove (2.4) in the right-hand pyramid P_+

$$(2.5) \quad 0 \leq x - x_0 < \delta = \min(\gamma, \alpha_0), |y_k - \dot{y}_k| \leq a_k - L(x - x_0) \quad (k = 1, \dots, n)$$

Put, for $0 \leq t < \delta$

$$M^i(t) = \max\{u^i(x_0 + t, Y) : Y \in P_t\}$$

$$N^i(t) = \max\{-u^i(x_0 + t, Y) : Y \in P_t\}$$

$$W(t) = \max_i \max\{|u^i(x_0 + t, Y)| : Y \in P_t\} = \|U(x_0 + t, \cdot)\|$$

where for $t \in [0, \delta)$ we have put $P_t = \{Y : (x_0 + t, Y) \in P_+\}$.

It is obvious that the inequalities (2.4) which are to be proved are equivalent to

$$(2.6) \quad W(t) \leq \omega(t, \eta) \quad \text{for } 0 \leq t < \delta.$$

By (2.2) we have

$$W(0) \leq \eta$$

and by Theorem 34.1 in [1] $W(t)$ is continuous in $[0, \delta)$. The inequality (2.6) will be proved if we show that the differential inequality

$$(2.7) \quad D_-W(t) \leq \sigma(t, W(t))$$

holds true in the set

$$(2.8) \quad E = \{t \in (0, \delta) : W(t) > \omega(t, \eta)\}$$

(see [1], § 14).

To prove (2.7) in (2.8) fix a $\tilde{t} \in E$; then we have

$$(2.9) \quad W(\tilde{t}) > \omega(\tilde{t}, \eta).$$

By Theorem 34.1 in [1] there is an index j and a point $\tilde{Y} \in P_{\tilde{t}}$ so that either

$$(2.10) \quad W(\tilde{t}) = M^j(\tilde{t}) = u^j(\tilde{t}, \tilde{Y}), D_-W(\tilde{t}) \leq D_-M^j(\tilde{t})$$

or

$$(2.11) \quad W(\tilde{t}) = N^j(\tilde{t}) = -u^j(\tilde{t}, \tilde{Y}), D_-W(\tilde{t}) \leq D_-N^j(\tilde{t}).$$

Suppose that, for instance, relations (2.10) hold true. By theorem 35.1 in [1] we have

$$(2.12) \quad D-M^i(\vec{i}) \leq u_x^i(x_0 + \vec{i}, \vec{Y}) - L \sum_{k=1}^n |u_{y_k}^i(x_0 + \vec{i}, \vec{Y})|.$$

On the other hand by (2.3) and the definition of $W(t)$ we get

$$\begin{aligned} u_x^i(x_0 + \vec{i}, \vec{Y}) &\leq \sigma(\vec{i}, \|U(x_0 + \vec{i}, \cdot)\|) + L \sum_{k=1}^n |u_{y_k}^i(x_0 + \vec{i}, \vec{Y})| \\ &= \sigma(\vec{i}, W(\vec{i})) + L \sum_{k=1}^n |u_{y_k}^i(x_0 + \vec{i}, \vec{Y})|. \end{aligned}$$

From (2.10), (2.12) and the last inequality it follows that

$$D-W(\vec{i}) \leq \sigma(\vec{i}, W(\vec{i})).$$

This completes the proof.

THEOREM 2.2. *Let the functions $U(x, Y) = (u^1(x, Y), \dots, u^m(x, Y))$ be of class \mathfrak{D} in the pyramid (2.1). Assume that*

$$(2.13) \quad U(x_0, Y) = 0$$

and that the inequalities

$$(2.14) \quad |u_x^i| \leq \sigma(|x - x_0|, \|U(x, \cdot)\|) + L \sum_{k=1}^n |u_{y_k}^i| \quad (i = 1, \dots, m)$$

are satisfied in the pyramid (2.1) for $x \neq x_0$ where $\sigma(t, y)$ satisfies Assumptions H_2 . Under these hypotheses we have

$$U(x, Y) \equiv 0$$

in the pyramid (2.1).

Proof. As in Theorem 2.1, it is sufficient to prove our theorem in the right-hand pyramid (2.5) (with $\delta = \gamma$). Put, for $0 \leq t < \delta$

$$\begin{aligned} M^i(t) &= \max \{u^i(x_0 + t, Y) : Y \in P_t\} \\ N^i(t) &= \max \{-u^i(x_0 + t, Y) : Y \in P_t\} \quad (i = 1, \dots, m) \\ W(t) &= \max_t \max \{|u^i(x_0 + t, Y)| : Y \in P_t\} \end{aligned}$$

Identities to be proved in the pyramid (2.5) are equivalent to

$$(2.15) \quad W(t) \equiv 0 \quad \text{for } t \in [0, \gamma].$$

By (2.13) we have

$$W(0) = 0$$

and by Theorem 34.1 in [1] $W(t)$ is continuous on $[0, \gamma)$. By the same theorem, for every $t \in (0, \gamma)$ there is an index j and a point $Y \in P_t$ such that either

$$(2.16) \quad W(t) = M^j(t) = u^j(x_0 + t, Y), \quad D_-W(t) \leq D^-M^j(t)$$

or

$$(2.17) \quad W(t) = N^j(t) = -u^j(x_0 + t, Y), \quad D_-W(t) \leq D^-N^j(t).$$

Suppose, for example, that for a $t \in (0, \gamma)$ relations (2.17) hold true. By theorem 35.1 in [1], we have

$$D_-N^j(t) \leq u_x^j(x_0 + t, Y) - L \sum_{k=1}^n |u_{y_k}^j(x_0 + t, Y)|.$$

As in the previous theorem from (2.14), the last inequality and (2.17) it follows that

$$D_-W(t) \leq \sigma(t, W(t)) \quad \text{for } t \in (0, \gamma).$$

Hence, by the second comparison theorem (see § 14 in [1]) we conclude that $W(t) \leq 0$ in $[0, \gamma)$ and since obviously $W(t) \geq 0$ we get (2.15), which completes the proof.

THEOREM 2.3. *Let the functions $U(x, Y) = (u^1(x, Y), \dots, u^m(x, Y))$ be of class \mathcal{D}_0 in the pyramid (2.1). Assume that*

$$(2.18) \quad U(x_0, Y) = U_x(x_0, Y) = 0$$

where $U_x(x, Y) = (u_x^1(x, Y), \dots, u_x^m(x, Y))$ and the inequalities (2.14) are satisfied in the pyramid (2.1) for $x \neq x_0$, where $\sigma(t, y)$ satisfies Assumptions H_3 .

Under these assumptions we have

$$U(x, Y) \equiv 0$$

in the pyramid (2.1).

Proof. Again we will prove our theorem in the right-hand pyramid (2.5) with $\delta = \gamma$. As in the previous theorem identity $U(x, Y) \equiv 0$ is equivalent with (2.15). By (2.18) we have

$$(2.19) \quad W(0) = 0.$$

Moreover, by Theorem 34.1 in [1] there is an index j such that

$$(2.20) \quad D^+W(0) \leq D^+M^j(0)$$

or

$$(2.21) \quad D^+W(0) \leq D^+N^j(0).$$

Suppose, for instance, that (2.20) holds true. On the other hand, by theorem 35.1 in [1] there is a point $Y_0 \in P_t$ such that

$$D^+W(0) \leq D^+M^j(0) \leq u_x^j(x_0, Y_0).$$

Hence, by (2.18), it follows that

$$(2.22) \quad D^+W(0) \leq 0.$$

As in the previous theorem we prove that

$$(2.23) \quad D_-W(t) \leq \sigma(t, W(t)) \quad \text{for } t \in (0, \gamma).$$

From (2.19), (2.22) and (2.23) we conclude, by the third comparison theorem (see § 14 in [1]), that $W(t) \leq 0$ for $t \in [0, \gamma)$ and consequently $W(t) = 0$.

3. Comparison theorems for overdetermined systems of partial differential-functional inequalities.

THEOREM 3.1. *Let the functions $U(X, Y) = (u^1(X, Y), \dots, u^m(X, Y)) = (u^1(x_1, \dots, x_p, y_1, \dots, y_n), \dots, u^m(x_1, \dots, x_p, y_1, \dots, y_n))$ be of class \mathcal{D} in the pyramid P (see 1). Suppose the initial inequalities*

$$(3.1) \quad |u^i(X_0, Y)| \leq \eta \quad (i = 1, \dots, m)$$

where $X_0 = (x_1, \dots, x_p)$, and the differential-functional inequalities

$$(3.2) \quad |u_{x_j}^i| \leq \sigma \left(\sum_{r=1}^p |x_r - \hat{x}_r|, \|U(X, \cdot)\| \right) + L \sum_{k=1}^n |u_{y_k}^i|$$

$$(i = 1, \dots, m; j = 1, \dots, p)$$

hold true in the pyramid P , where the function $\sigma(t, y)$ satisfies Assumptions H_1 . Let $\omega(t, \eta)$ be the right-hand maximum solution of (1.1) through $(0, \eta)$, defined in an interval

$$0 \leq t < \alpha_0.$$

Under these assumptions we have

$$|u^i(X, Y)| \leq \omega \left(\sum_{r=1}^p |x_r - \hat{x}_r|; \eta \right) \quad (i = 1, \dots, m)$$

in the pyramid

$$(3.3) \quad \sum_{j=i}^p |x_j - \hat{x}_j| < \min(\gamma, \alpha_0), |y_k - \hat{y}_k| \leq a_k - L \sum_{j=1}^p |x_j - \hat{x}_j| \quad (k = 1, \dots, n).$$

Proof. We will use Mayer's transformation

$$X = X_0 + \Lambda x,$$

where $\Lambda = (\lambda_1, \dots, \lambda_p)$ in order to reduce our theorem to theorem 2.1. For $\Lambda = (\lambda_1, \dots, \lambda_p)$ consider the ordinary equation

$$\frac{dy}{dt} = \lambda \sigma(\lambda t, y)$$

where $\sigma(\lambda t, y)$ satisfies Assumptions H_1 and $\lambda = \sum_{j=1}^p |\lambda_j|$. By Theorem (36.1) in [1] we know that $\omega(\lambda t, \eta)$ is its right-hand maximum solution through $(0, \eta)$ in the interval $[0, \alpha_0/\lambda]$. In particular, for $\lambda < \alpha_0$ we have

$$(3.4) \quad \frac{\alpha_0}{\lambda} > 1.$$

Suppose that

$$(3.5) \quad \lambda < \min(\gamma, \alpha_0)$$

and denote

$$\tilde{U}(x, Y; \Lambda) = U(X_0 + \Lambda x, Y).$$

For Λ satisfying (3.5) $\tilde{U}(x, Y; \Lambda)$ is of class \mathfrak{D} in the pyramid

$$(3.6) \quad |x| < \frac{\gamma}{\lambda}, \quad |y_k - \check{y}_k| \leq \alpha_k - \lambda|x| \quad (k = 1, \dots, n)$$

where

$$(3.7) \quad \frac{\gamma}{\lambda} > 1.$$

By (3.1) and (3.2) we have

$$|\tilde{u}^i(0, Y; \Lambda)| \leq \eta \quad (i = 1, \dots, m)$$

and

$$\begin{aligned} |\tilde{u}_x^i(x, Y; \Lambda)| &\leq \sum_{j=1}^p |\lambda_j| |u_{x_j}^i(X_0 + \Lambda x, Y)| \leq \sum_{j=1}^p |\lambda_j| \sigma\left(\sum_{r=1}^p |\lambda_r x|, \|U(X_0 + \Lambda x, \cdot)\|\right) + \\ &\quad + L \sum_{j=1}^p |\lambda_j| \sum_{k=1}^n |u_{y_k}^i(X_0 + \Lambda x, Y)| \\ &= \lambda \sigma(\lambda|x|, \|\tilde{U}(x, \cdot; \Lambda)\|) + L\lambda \sum_{k=1}^n |\tilde{u}_{y_k}^i(x, Y; \Lambda)|. \end{aligned}$$

Hence, by Theorem 2.1, we have

$$(3.8) \quad |\tilde{u}^i(x, Y; \Lambda)| \leq \omega(\lambda|x|, \eta) \quad (i = 1, \dots, m)$$

in the pyramid (3.6) for

$$|x| < \min\left(\frac{\gamma}{\lambda}, \frac{\alpha_0}{\lambda}\right).$$

From (3.4) and (3.7) it follows that

$$\min\left(\frac{\gamma}{\lambda}, \frac{\alpha_0}{\lambda}\right) > 1$$

and we can put $x = 1$ into (3.8). We get the inequalities

$$|\tilde{u}^i(1, Y; A)| \leq \omega(\lambda, \eta)$$

for λ satisfying (3.5). Hence for any point (X, Y) of the pyramid (3.3) and for $A = X - X_0$ we have

$$|u^i(X, Y)| = |\tilde{u}^i(1, Y; X - X_0)| \leq \omega\left(\sum_{j=1}^p |x_j - \hat{x}_j|; \eta\right) \quad (i = 1, \dots, m)$$

which completes the proof.

Similarly, using Theorems 36.2 and 36.3 in [1] and (2.2), (2.3) of this paper we can prove the following two theorems:

THEOREM 3.2. *Let the functions $U(X, Y) = (u^1(X, Y), \dots, u^m(X, Y))$ be of class \mathcal{D} in the pyramid P . Suppose that*

$$U(X_0, Y) = 0$$

and

$$(3.9) \quad |u_{x_j}^i| \leq \sigma\left(\sum_{r=1}^p |x_r - \hat{x}_r|, \|U(X, \cdot)\|\right) + L \sum_{k=1}^n |u_{y_k}^i| \quad \text{for } X \neq X_0$$

$$(i = 1, \dots, m; j = 1, \dots, p)$$

in the pyramid P , where $\sigma(t, y)$ satisfies Assumptions H_2 . Under these assumptions we have

$$U(X, Y) = 0$$

in the pyramid P .

THEOREM 3.3. *Let the functions $U(X, Y)$ be of class \mathcal{D}_0 in the pyramid P . Suppose that*

$$U(X_0, Y) = U_{x_j}(X_0, Y) = 0 \quad (j = 1, \dots, p)$$

where $U_{x_j} = (u_{x_j}^1, \dots, u_{x_j}^m)$ and that inequalities (3.9) hold true in the pyramid P with $\sigma(t, y)$ satisfying Assumptions H_3 . Then we have

$$U(X, Y) = 0$$

in the pyramid P .

4. Estimates of the solution.

THEOREM 4.1. *Let the right-hand members*

$$f_j^i(X, Y, U, Q, Z)$$

$$= f_j^i(x_1, \dots, x_p, y_1, \dots, y_n, u^1, \dots, u^m, q_1, \dots, q_n, z^1, \dots, z^m)$$

$$(i = 1, \dots, m; j = 1, \dots, p)$$

of system (1) be defined for $(X, Y) \in P$, U, Q arbitrary and $Z \in C_m(P_X)$ (see 1). Suppose that

$$(4.1) \quad |f_j^i(X, Y, U, Q, Z)| \leq \sigma \left(\sum_{r=1}^p |x_r - \hat{x}_r|, \max_l \{|u^l|, \|Z\|\} \right) + L \sum_{k=1}^n |q_k|$$

$$(i = 1, \dots, m; j = 1, \dots, p)$$

where $\sigma(t, y)$ satisfies Assumptions H_1 . Let $\omega(t, \eta)$ be the right-hand maximum solution of (1.1) through $(0, \eta)$, defined in the interval $[0, \alpha_0)$. Let $U(X, Y)$ be a solution of system (1) of class \mathcal{D} in the pyramid P and satisfying the initial inequalities

$$(4.2) \quad |u^i(X_0, Y)| \leq \eta \quad (i = 1, \dots, m).$$

Under these assumptions we have

$$(4.3) \quad |u^i(X, Y)| \leq \omega \left(\sum_{r=1}^p |x_r - \hat{x}_r|, \eta \right)$$

in the pyramid

$$(4.4) \quad \sum_{r=1}^p |x_r - \hat{x}_r| < \min(\gamma, \alpha_0), |y_k - \hat{y}_k| \leq \alpha_k - L \sum_{r=1}^p |x_r - \hat{x}_r| \quad (k = 1, \dots, n).$$

Proof. By (4.1) and by the definition of $\|Z\|$ we have

$$\begin{aligned} |u_{x_j}^i(X, Y)| &= |f_j^i(X, Y, U(X, Y), u_Y^i(X, Y), U(X, \cdot))| \\ &\leq \sigma \left(\sum_{r=1}^p |x_r - \hat{x}_r|, \max_l \{|u^l(X, Y)|, \|U(X, \cdot)\|\} \right) + L \sum_{k=1}^n |u_{y_k}^i(X, Y)| \\ &= \sigma \left(\sum_{r=1}^p |x_r - \hat{x}_r|, \|U(X, \cdot)\| \right) + L \sum_{k=1}^n |u_{y_k}^i(X, Y)| \end{aligned}$$

and the function $U(X, Y)$ satisfies all the assumptions of Theorem 3.1. Hence the inequalities (4.3) hold true in the pyramid (4.4).

5. Estimates of the difference between two solutions.

THEOREM 5.1. Let $f_j^i(X, Y, U, Q, Z)$ the right-hand members of system (1) and $g_j^i(X, Y, U, Q, Z)$ of system

$$(5.1) \quad u_{x_j}^i = g_j^i(X, Y, U(X, Y), u_Y^i(X, Y), U(X, \cdot)) \quad (i = 1, \dots, m; j = 1, \dots, p)$$

be defined for $(X, Y) \in P$, U, Q arbitrary and $Z \in C_m(P_X)$, and satisfy the inequalities

$$(5.2) \quad |f_j^i(X, Y, U, Q, Z) - g_j^i(X, Y, \tilde{Y}, \tilde{Q}, \tilde{Z})|$$

$$\leq \sigma \left(\sum_{r=1}^p |x_r - \hat{x}_r|, \max_l \{|u^l - \tilde{u}^l|, \|Z - \tilde{Z}\|\} \right) + L \sum_{k=1}^n |q_k - \tilde{q}_k|$$

$$(i = 1, \dots, m; j = 1, \dots, p)$$

where $\sigma(t, y)$ satisfies Assumptions H_1 . Let $\omega(t, \eta)$ be the right-hand maximum solution of (1.1) through $(0, \eta)$ defined in the interval $[0, \alpha_0)$. Suppose that $U(X, Y) = (u^1(X, Y), \dots, u^m(X, Y))$ and $V(X, Y) = (v^1(X, Y), \dots, v^m(X, Y))$ are two solutions of systems (1) and (5.1) respectively, of class \mathfrak{D} in the pyramid P and satisfying the initial inequalities

$$|u^i(X_0, Y) - v^i(X_0, Y)| \leq \eta \quad (i = 1, \dots, m).$$

Under these assumptions we have

$$(5.3) \quad |u^i(X, Y) - v^i(X, Y)| \leq \omega\left(\sum_{r=1}^p |x_r - \hat{x}_r|, \eta\right) \quad (i = 1, \dots, m)$$

in the pyramid (4.4).

Proof. Function $\tilde{U}(X, Y) = U(X, Y) - V(X, Y)$ satisfies by (5.2) the inequalities (3.2) of Theorem 3.1

$$\begin{aligned} |\tilde{u}_{x_j}^i| &= |u_{x_j}^i - v_{x_j}^i| = |f_j^i(X, Y, U(X, Y), u_Y^i(X, Y), U(X, \cdot)) - \\ &\quad - g_j^i(X, Y, V(X, Y), v_Y^i(X, Y), V(X, \cdot))| \\ &\leq \sigma\left(\sum_{r=1}^p |x_r - \hat{x}_r|, \max_l (|u^l - v^l|, \|U(X, \cdot) - V(X, \cdot)\|)\right) + \\ &\quad + L \sum_{k=1}^n |u_{y_k}^i - v_{y_k}^i| = \sigma\left(\sum_{r=1}^p (|x_r - \hat{x}_r|, \|\tilde{U}(X, \cdot)\|)\right) + L \sum_{k=1}^n |\tilde{u}_{y_k}^i| \\ &\quad (i = 1, \dots, m; j = 1, \dots, p) \end{aligned}$$

and because it satisfies all other assumptions of this theorem, the inequalities (5.3) hold true.

6. Uniqueness criteria

THEOREM 6.1. Let the right-hand member $f_j^i(X, Y, U, Q, Z)$ of system (1) be defined for $(X, Y) \in P$, U, Q arbitrary and $Z \in C_m(P_X)$ and satisfy the inequalities

$$(6.1) \quad |f_j^i(X, Y, U, Q, Z) - f_j^i(X, Y, \tilde{U}, \tilde{Q}, \tilde{Z})| \leq \sigma\left(\sum_{r=1}^p |x_r - \hat{x}_r|, \max_l (|u^l - \tilde{u}^l|, \|Z - \tilde{Z}\|)\right) + L \sum_{k=1}^n |q_k - \tilde{q}_k| \quad (i = 1, \dots, m; j = 1, \dots, p)$$

where $\sigma(t, y)$ satisfies Assumptions H_2 (Assumptions H_3). Under these assumptions the Cauchy problem for system (1) with initial data

$$(6.2) \quad u^i(X_0, Y) = \varphi^i(Y) \quad i = 1, \dots, m$$

admits at most one solution of class \mathfrak{D} (class \mathfrak{D}_0) in the pyramid P .

Proof. Let $U(X, Y) = (u^1(X, Y), \dots, u^m(X, Y))$ and

$$V(X, Y) = (v^1(X, Y), \dots, v^m(X, Y))$$

be two solutions of problem (1), (6.2) of class \mathfrak{D} (of class \mathfrak{D}_0) and

$$\tilde{U}(X, Y) = U(X, Y) - V(X, Y).$$

Then we have

$$(6.3) \quad \tilde{U}(X_0, Y) = U(X_0, Y) - V(X_0, Y) = 0$$

and by (6.1)

$$\begin{aligned} |\tilde{u}_{x_j}^i| &= |f_j^i(X, Y, U, u_{Y'}^i, U(X, \cdot)) - f_j^i(X, Y, V, v_{Y'}^i, V(X, \cdot))| \\ &\leq \sigma \left(\sum_{r=1}^p |x_r - \hat{x}_r|, \max_l (|u^l - v^l|, \|U(X, \cdot) - V(X, \cdot)\|) \right) + \\ &+ L \sum_{k=1}^n |u_{y_k}^i - v_{y_k}^i| = \sigma \left(\sum_{r=1}^k |x_r - \hat{x}_r|, \|\tilde{U}(X, \cdot)\| \right) + L \sum_{k=1}^r |\tilde{u}_{y_k}^i| \\ &\quad (i = 1, \dots, m; j = 1, \dots, p). \end{aligned}$$

Moreover by (6.3)

$$U_{y_k}(X_0, Y) \equiv V_{y_k}(X_0, Y)$$

hence

$$\begin{aligned} \tilde{u}_{x_j}^i(X_0, Y) &= u_{x_j}^i(X_0, Y) - v_{x_j}^i(X_0, Y) \\ &= f_j^i(X_0, Y, U(X_0, Y), u_{Y'}^i(X_0, Y), U(X, \cdot)) - \\ &- f_j^i(X_0, Y, V(X_0, Y), v_{Y'}^i(X_0, Y), V(X_0, \cdot)) = 0 \\ &\quad (i = 1, \dots, m, j = 1, \dots, p). \end{aligned}$$

Hence, if $\sigma(t, y)$ satisfies Assumptions H_2 (Assumptions H_3), $\tilde{U}(X, Y)$ satisfies all the assumptions of Theorem 3.2 (Theorem 3.3) and we have

$$U(X, Y) \equiv 0$$

in the pyramid P .

7. Continuous dependence of the solution on the initial data and on the right-hand sides of the system

THEOREM 7.1. *Let the right-hand members $f_j^i(X, Y, U, Q, Z)$ of system (1) and $g_j^i(X, Y, U, Q, Z)$ of system (5.1) be defined for $(X, Y) \in P$, U, Q — arbitrary and $Z \in C_m(P_X)$. Let the functions $f_j^i(X, Y, U, Q, Z)$ satisfy the inequalities (6.1) where $\sigma(t, y)$ satisfies Assumptions H_1 . Suppose that*

$$\sigma(t, 0) \equiv 0$$

