

Attempted generalization of the idea of a tensor product

by A. L. DAWIDOWICZ

0. Introduction. In this paper I shall try to generalize the definition of a tensor product for any concrete category and to formulate the theorems known in the category of modules for any category. I wish to express my gratitude to Dr. E. Tutaj, who helped me in the section on topology, Dr. B. Grell, thanks to whom the paper has been accomplished, Mr K. M. Werber, thanks to whom the idea of these concepts was conceived.

Terminology

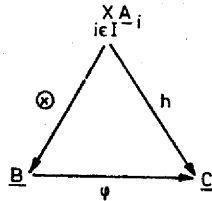
- $f: A \subset \rightarrow B$ — f is an injection A in B
 $f: A \twoheadrightarrow B$ — f is a surjection A on B
 $\exists!$ — there exists one and only one
 $\times_{i \in I} A_i$ — a Cartesian product
 $A = (\underline{A}, \text{str } A)$ — an object of concrete category
 \underline{A} — the underlying set of A
 $\text{str } A$ — the structure of A
 $\mathcal{C}(A, B)$ — the set of morphisms $A \rightarrow B$ in category \mathcal{C} for $f \in \mathcal{C}(A, B)$
 we note also $f: A \xrightarrow{\mathcal{C}} B$ or $f: A \rightarrow B$
 $\mathcal{A}_\nu[N]$ — the free ν -algebra spread on N (the ν -algebra of terms on N)
 $N = \{0, 1, 2, \dots\}$
 \mathcal{A}_ν — the category of ν -algebras, defined by the relation where
 $\nu: S \rightarrow N, \tau \in (\mathcal{A}_\nu[N])^2$
 $S_i = \{k: \nu(k) = i\}$
 $\text{top } A$ — the family of open subsets of A , i.e. $\text{str } A = \text{top } A$ for
 $A \in \text{obj } \mathcal{H}$, where \mathcal{H} is the category of topological spaces.

1. Fundamental concepts. Let \mathcal{C} be any concrete category, I be any set. Let $A: I \rightarrow \text{obj } \mathcal{C}$ $i \in I$ $x \in \times_{j \in I \setminus \{i\}} \underline{A}_j$. We define $\text{in}^x = \text{in}_i^x: \underline{A}_i \rightarrow \times_{j \in I} \underline{A}_j$, by the formula $\text{in}^x(y) = x \cup \{(i, y)\}$.

Definition: We call a map $h: \times_{i \in I} \underline{A}_i \rightarrow \underline{B}$ ($B \in \text{obj } \mathcal{C}$) the I -morphism (polymorphism) iff $\forall i \in I \forall x \in \times_{j \in I \setminus \{i\}} \underline{A}_j$ $h^x = h \circ \text{in}^x \in \mathcal{C}(A_i, B)$.

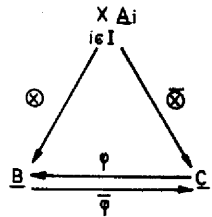
Definition: A universal I -morphism $\times_{i \in I} \underline{A}_i$ in an object of category \mathcal{C} is called a tensor product of the family $\{A_i\}_{i \in I}$ of the objects of the category \mathcal{C} , i.e. \otimes is a tensor product of a family $\{A_i\}_{i \in I}$ iff

- 1° $\otimes: \times_{i \in I} \underline{A}_i \rightarrow \underline{B}$ ($B \stackrel{\text{def}}{=} \otimes_{i \in I} A_i = \otimes A$)
- 2° \otimes is an I -morphism
- 3° $\forall h: \times_{i \in I} \underline{A}_i \rightarrow \underline{C}$ if h is an I -morphism, then $\exists! \varphi \in \mathcal{C}(B, C): \varphi \circ \otimes = h$



THEOREM: A tensor product is unequivocally determined exact to an isomorphism.

Proof: Suppose two different tensor products \otimes and $\overline{\otimes}$ exist, $\otimes: \times_{i \in I} \underline{A}_i \rightarrow \underline{B}$ and $\overline{\otimes}: \times_{i \in I} \underline{A}_i \rightarrow \underline{C}$.

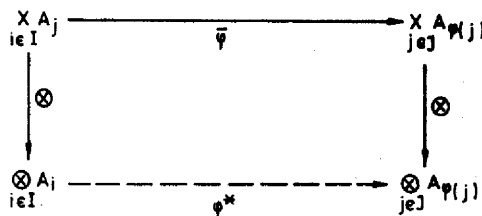


\otimes is an I -morphism, hence $\exists! \varphi \in \mathcal{C}(C, B): \varphi \circ \overline{\otimes} = \otimes$
 $\overline{\otimes}$ is an I -morphism, hence $\exists! \overline{\varphi} \in \mathcal{C}(B, C): \overline{\varphi} \circ \otimes = \overline{\otimes}$
 $\overline{\varphi} \circ \varphi \in \mathcal{C}(C, C)$ and $(\overline{\varphi} \circ \varphi) \circ \overline{\otimes} = \otimes$ and id_C satisfy the same properties so $\overline{\varphi} \circ \varphi = \text{id}_C$ and analogously $\varphi \circ \overline{\varphi} = \text{id}_B$, hence $\varphi \in \text{iso } \mathcal{C}$

2. Problem of associativity and commutativity. Let be $\varphi: J \hookrightarrow I$

THEOREM: If $\exists \otimes_{i \in I} A_i, \exists \otimes_{j \in J} A_{\varphi(j)}$, then $\exists! \varphi^*: \otimes_{i \in I} A_i \rightarrow \otimes_{j \in J} A_{\varphi(j)}: \varphi^*(\otimes(x)) = \otimes_{j \in J} x_{\varphi(j)}$.

Proof: $\exists \overline{\varphi}: \times_{i \in I} \underline{A}_i \rightarrow \times_{j \in J} \underline{A}_{\varphi(j)}$ defined by formula $\overline{\varphi}(x) = x \circ \varphi$ i.e. $[\overline{\varphi}(x)]_j = x_{\varphi(j)}$



For proving the existence and uniqueness of φ^* we must prove, that $\otimes \circ \bar{\varphi}$ is an I -morphism.

Let $x \in \times_{j \in I \setminus \{i\}} \underline{A}_j$ in $\text{in}^x: \underline{A}_i \rightarrow \times_{j \in I} \underline{A}_j$ in $\text{in}^x(y) = x \cup \{i; y\}$,

$\text{in}^x(y) \circ \varphi = x \circ \varphi \cup \{(\varphi^{-1}(i); y)\} = \text{in}^{x \circ \varphi}(y)$

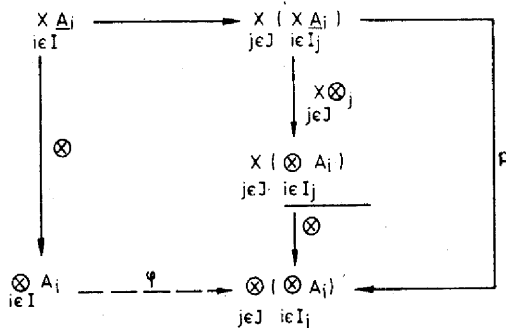
$\otimes \circ \bar{\varphi} \circ \text{in}^x(y) = \otimes \circ \text{in}^x(y) \circ \varphi = \otimes \circ \text{in}^{x \circ \varphi}(y)$.

Corollary: If $\varphi: J \hookrightarrow I$ is a bijection, then $\otimes_{i \in I} A_i \cong \otimes_{j \in J} A_{\varphi(j)}$.

THEOREM: If $I = \bigcup_{j \in I} I_j$ -separable union and the category \mathcal{C} is with tensor products, then

$$\exists! \varphi: \otimes_{i \in I} A_i \rightarrow \otimes_{j \in J} \otimes_{i \in I_j} A_i: \varphi(\otimes_{i \in I} x_i) = \otimes_{i \in I} \otimes_{j \in J} x_{i,j}$$

Proof:



where $\otimes_j: \times_{i \in I_j} A_i \rightarrow \otimes_{i \in I_j} A_i$

$$p(x) = \otimes_{j \in J} \otimes_{i \in I_j} x_i$$

Let now $x \in \times_{i \in I \setminus \{i_0\}} \underline{A}_i$ $x(j) = x|_{I_j}$ $i_0 \in I_{j_0}$

$\text{in}^x(y) = x \cup \{(i_0; y)\} = \bigcup_{j \in J \setminus \{j_0\}} x(j) \cup x|_{I_{j_0}} \cup \{(i_0; y)\}$

$= \bigcup_{j \in J \setminus \{j_0\}} x(j) \cup \text{in}^{x(j_0)}(y) = ((\text{in}^{x(j)})_{j \in J \setminus \{j_0\}} \cdot \text{in}^{x(j_0)})(y)$

then $p \circ \text{in}^x = \underbrace{\otimes \circ (\text{in}^{\otimes_j x(j)})_{j \in J \setminus \{j_0\}}}_{\text{morphism}} \circ \underbrace{\otimes_{j_0} \text{in}^{x(j_0)}}_{\text{morphism}}$

Then p is an I -morphism, hence $\exists! \varphi$ such that $\varphi(\otimes x) = p(x)$.

Definition: A tensor product is said to be associative, iff φ (from the preceding theorem) is an isomorphism.

3. Tensor products in $\mathcal{A}_{\nu r}$

THEOREM: The category $\mathcal{A}_{\nu r}$ is with tensor products.

Proof: Let $A: I \ni i \rightarrow (A_i; \alpha^i) = A_i \in \text{obj } \mathcal{A}_{\nu r}$. I spread a free νr -algebra $B = (B, \beta) = \mathcal{A}_{\nu r}[\times_{i \in I} \underline{A}_i]$ on $\times_{i \in I} \underline{A}_i$.