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Note on a functional inequality with the n-th iterate of the unknown function

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In this paper we shall deal with solutions of the inequality

$$\psi^n(x) \geqslant g(x) \tag{1}$$

in the class $C[a, b]$ of functions ψ: $[a, b] \to [a, b]$ continuous in the interval $[a, b]$. ψ^n denotes here the n-th interate of the function ψ. We shall assume the following hypothesis
(H) The function g is strictly increasing and continuous in the interval $[a, b]$,

$$g(a) = a, \; g(b) = b, \; g(x) > x \quad \text{for } x \in (a, b).$$

We proved in [1], inter alia, the following

THEOREM 1. *Let the function g fulfil the hypothesis (H). Let a function ψ satisfy the conditions*

$$\psi \in C[a, b], \; \psi(x) > x \quad \textit{for } x \in (a, b), \; \psi(a) = a, \; \psi(b) = b,$$

and let there exist real numbers $\alpha, \beta \in (0, 1)$, and a', b' such that

$$\psi(x) > \alpha g(x) + (1-\alpha)x \quad \textit{for } x \in (a, a'), \tag{2}$$

$$\psi(x) > \beta g(x) + (1-\beta)x \quad \textit{for } x \in (b', b). \tag{3}$$

If there exists a positive number s such that

$$g(y) - g(x) \geqslant s(y-x) \tag{4}$$

for $y > x$ and $x, y \in (a, a')$ or $x, y \in (b', b)$, then there is a positive integer n such that the inequality (1) *holds for $x \in [a, b]$.*

Remark 1. Conditions (2) and (3) are equivalent to the following:

$$\liminf_{x \to a+} \frac{\psi(x) - x}{g(x) - x} > 0 \tag{2'}$$

$$\liminf_{x \to b-} \frac{\psi(x) - x}{g(x) - x} > 0 \tag{3'}$$

The aim of this paper is to show that in a certain class of functions the conditions (2′) and (3′) are also necessary for the function $\psi \in C[a, b]$ to be a solution of (1) for a positive integer n. Namely we shall prove

THEOREM 2. *Let the function g fulfil hypothesis (H) and let a continuous function $\psi\colon [a, b] \to [a, b]$ fulfil the inequality*

$$\psi(x) > x \quad \textit{for } x \in (a, b) . \tag{5}$$

a) *If there exists a number $A > 0$ such that*

$$g(y) - y \leqslant A(g(x) - x) \tag{6}$$

for $x, y \in [a, a') \subset [a, b]$, $g(x) > y > x$
and

$$\lim_{x \to a+} \frac{\psi(x) - x}{g(x) - x} = 0 , \tag{7}$$

then for every positive integer n there exists a number $a_n \in (a, a')$ such that

$$\psi^n(x) > g(x) \quad \textit{for } x \in (a, a_n) .$$

b) *If there exists a number $B > 0$ such that*

$$g(y) - y \leqslant B(g(x) - x) \tag{8}$$

for $x, y \in (b', b] \subset (a, b]$, $y > x$
and

$$\lim_{x \to b-} \frac{\psi(x) - x}{g(x) - x} = 0 , \tag{9}$$

then for every positive integer n there exists a number $b_n \in (b', b)$ such that

$$\psi^n(x) > g(x) \quad \textit{for } x \in (b_n, b) .$$

Proof. We shall prove part a) of the theorem. First we shall prove by induction the following equality

$$\lim_{x \to a+} \frac{\psi^n(x) - x}{g(x) - x} = 0 \quad \text{for } n = 1, 2, \ldots \tag{10}$$

For $n = 1$ the condition (10) reduces to (7).
Let the condition (10) be fulfilled for a $k \geqslant 1$. We have

$$\frac{\psi^{k+1}(x) - x}{g(x) - x} = \frac{\psi^k(y) - y}{g(y) - y} \cdot \frac{g(y) - y}{g(x) - x} + \frac{\psi(x) - x}{g(x) - x} \tag{11}$$

where we put $y := \psi(x)$. By (7) and (5) we have $g(x) > y > x$ in a neighbourhood $(a, a+\delta) \subset [a, a')$ of the point a, and from (6) we get

$$0 \leqslant \frac{g(y) - y}{g(x) - x} \leqslant A \quad \text{for } x \in (a, a+\delta) . \tag{12}$$

It follows from (7) that $\lim_{x\to a+}(\psi(x)-x)=0$ so that $\psi(a)=a$ since the ψ is continuous at a. Consequently if $x\to a$, then $y\to a$ and passing to the limit in (11) we get by the inductive hypothesis, (12) and (7) the equality

$$\lim_{x\to a+}\frac{\psi^{k+1}(x)-x}{g(x)-x}=0$$

and the inductive proof of (10) is completed.

Let n be a positive integer. It follows from (10) that there exists a number $a_n\in[a, a')$, for which

$$\frac{\psi^n(x)-x}{g(x)-x}<1 \qquad \text{for } x\in(a, a_n),$$

whence

$$\psi^n(x)<g(x) \qquad \text{for } x\in(a, a_n),$$

which ends the proof of the theorem in case a).

Case b) may be proved in a similar manner.

Remark 2. Assumption (6) is fulfilled if there exist a $K>0$ and an $a'>a$ such that

$$\frac{g(y)-g(x)}{y-x}<K$$

for $x, y\in[a, a')$, $x<y<g(x)$, i.e. it would also be enough to assume that the right derivative of the function g is bounded from above in the right neighbourhood of the point a. A similar remark is valid for case b) of the theorem (assumption (8)).

Reference

[1] E. Turdza, *On a Functional Inequality with the n-th Iterate of the Unknown Function*, Zeszyty Naukowe UJ, Prace Matematyczne 16, (1974), 189—194.