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On Gluskin's class of Matrix Algebras

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Assume K to be an Euclidean ring and $M(n, K)$ the set of all square matrice of order n with elements from K. The transvection matrix E_{ik} from $M(n, K$ has the element (i, k) equal to 1 and the remaining elements zero. Let A be an algebra over K with basis consisting of a set S *) of matrices E_{ik}. The full matrix algebra $M(n, K)$, the algebra of all triangular matrices $T(n, K)$, direct sums of full matrix algebras, and semi-reducible algebras are examples of those defined above.

L. M. Gluskin in his paper [2] makes the assumption that all the matrices E_{ii} belong to A and then finds all the automorphisms of the multiplicative semigroup A_M of A. He also shows that every automorphism φ of A_M is an automorphism of the algebra A itself provided A does not contain a simple factor of dimension 1.

1. First we study how the larger class of matrix algebras generated by transvection matrices is related to that considered by Gluskin (for simplicity let them be called "Gluskin's algebras"). We shall prove the following.

THEOREM 1. *The algebra A generated by transvection matrices is a Gluskin algebra if and only if A contains a non-singular matrix.*

In the proof we use the graph representation of A which has been introduced and worked out in the author's dissertation [1] (in preparation). Namely, the basis S of A can be uniquely represented by a directed graph $G(S)$ whose points are numbers $1, 2, \ldots, n$ and edges are images under the correspondence $E_{ik} \leftrightarrow (i, k)$ the last being the directed edge from i to k. In particular E_{ii} is represented by the loop (i, i). The following statements can be easily justified:

Statement 1. $G(S)$ contains a cycle **) $(k_1, \ldots, k_r)$ $r \geqslant 1$ if and only if all the matrices $E_{k_i k_j}$ $(i, j = 1, \ldots, r)$ belong to S (in particular all the idempotent transvection matrices $E_{k_i k_i}$). This statement follows from the fact that the set S is closed under matrix multiplication, hence if there is a walk from k_i to k_j in $G(S)$ then there must be also the edge $E_{k_i k_j}$ in $G(S)$ as well. Obviously, in a cycle any two of its points can be joined by a directed walk.

*) Observe that S is a semigroup under matrix multiplication.

**) i.e. a closed directed walk; the loop (i, i) in particular.

Statement 2. A is a Gluskin's algebra iff any point of $G(S)$ is lying in a cycle.

In fact, E_{ii} belong to S if $G(S)$ contains the loop (i, i) but this takes place if i is a point of a cycle.

Proof of the Theorem 1. ($\Rightarrow$). If A includes the matrices E_{ii} $(i = 1, \ldots, n)$ then A has also the unit matrix $E_{11}+\ldots+E_{nn}$. ($\Leftarrow$). Let A have a non-singular matrix $X = (x_{ij})$. Then $\det X \sum_{(j_1,\ldots,j_n)} sgn(j_1 \ldots j_n) X_{1j_1} \ldots X_{nj_n} \neq 0$. Hence at least product $X_{1j_1}, \ldots, X_{nj_n}$ is different from zero and consequently all the coefficients $X_{1j_1}, \ldots, X_{nj_n} \neq 0$. Thus the matrices $E_{1j_1}, \ldots, E_{nj_n}$ must be in the basis S. The permutation $(j_1, \ldots, j_n)$ decomposes into cycles, which means that any point $1, 2, \ldots, n$ belongs to a cycle from the permutation $(1, 2, \ldots, n) \to (j_1, \ldots, j_n)$ and consequently to the corresponding cycle in $G(S)$. In virtue of statement 2 this fact completes the proof.

Now we shall prove the following structural theorem.

THEOREM 2. *If A is an arbitrary algebra generated by transvection matrices then either*

(i) A is nilpotent, which holds if A has no idempotent matrix E_{ii}, or

(ii) $A = A_1 + N + B$ (vector space direct sum), where A_1 is a Gluskin algebra, N is a nilpotent subalgebra, $B^2 \subset A_1 + N$ and if $E_{ij} \in B$ then E_{ji} is not in A.

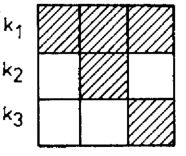
Proof. If A is nilpotent then obviously it does not contain any idempotent matrix $X = X^2$. Conversely, if A is free from all E_{ii}'s then the graph $G(S)$ has no loops and consequently has no cycle. It has been proved in the thesis [1] that in such a case the algebra A is nilpotent (i.e. $A^k = 0$ for some integer $k > 0$).

Assume now that A contains some idempotents $E_{i_1i_1}, \ldots, E_{i_ri_r}$ $(r > 0)$. Up to a permutation of indices we may admit that these are the first r idempotents $E_{11}, \ldots, E_{rr}$. Let A_1 be the subalgebra of A spanned by all the matrices from S, both the subindices of which are from the interval $[1, r]$; N the subalgebra spanned on that part of S of which the matrices have both subindices from $[r+1, n]$ and B the vector subspace spanned on the remaining transvection matrices from the basis S. Then, clearly A is the direct sum of its vector subspaces A_1, N and B. As A_1 contains all the matrices $E_{11}, \ldots, E_{rr}$, it is a Gluskin algebra. Similarly, in view of point (i) N is nilpotent since it has no idempotent transvection matrix. To complete the proof assume $E_{ij} \in B$, hence either i is not greater than r and j not less than r or vice versa. If E_{ji} were in A then both E_{ii} as well as E_{jj} would be in A; then E_{jj} would be in N (or E_{ii}, respectively) which contradicts the definition of N. The inclusion $B^2 \subset A_1 + N$ is clear.

Let us call a cycle in $G(S)$ maximal if it is not contained in a cycle having more points. Obviously, any two different maximal cycles C_1 and C_2 are either disconnected (there is no walk joining them) or are connected by walks all going in the same direction, say from points of C_1 to points of C_2. (Note that if C_1 is connected to C_2 then any points of C_1 is connected by a walk to any point of C_2). We shall use the notation $C_1 \succ C_2$ (C_1 preceeds C_2) in such a case.

By a permutation of indices we can obtain that the indices within each maximal cycle are consecutive. By Gluskin's assumption each point from $1, \ldots, n$

is on a cycle, hence in a maximal cycle. Let $C_1, \dots, C_r$ be the complete list of maximal cycles. We may assume that C_1 has points $1, \dots, k_1$; C_2 the points $k_1+1, \dots, k_1+k_2$ and so on, where $k_1, k_2, \dots$ are the number of points in $C_1, C_2, \dots$ For instance, if $r = 3$, $C_1 \succ C_2$ and $C_1 \succ C_3$ then the matrices from A will be of the form



(There may be arbitrary elements in the shaded boxes). Let $\alpha, \beta, \dots$ run from 1 to r. Each matrix $X \epsilon A$ can be considered as a block matrix $(X_{\alpha\beta})$ according to the decomposition into cycles C_α ($\alpha = 1, \dots, r$). Now, if $X_{\alpha\beta} \neq 0$ then it follows that $X_{\beta\alpha} = 0$, and this implies that there is no cycle of more than one point in the graph representing the transvection matrices $E_{\alpha\beta}$. As was shown in the thesis, if this is the case then the algebra spanned on all matrices $E_{\alpha\beta}$ is solvable and can be brought to an upper triangular form by a certain permutation of indices $1, \dots, r$. Thus we may conclude the following:

Proposition. *Any Gluskin algebra can be transformed by a permutation of indices into an upper quasi-triangular form with block-submatrices $X_{\alpha\beta}$ which are either zero or full* ***); *in particular $X_{\alpha\alpha}$, s are full matrix algebras.*

The transformation of the algebra A by permutation of indices is isomorphic. In fact, the map $(X_{ij}) \to (X_{\pi(i)\pi(j)})$ can be expressed in matrix form as $X \to P^{-1}XP$, where $P = (p_{ij})$ is the permutation matrix defined by

$$p_{ij} = \begin{cases} 1 & \text{if } j = \pi(i) \\ 0 & \text{otherwise} \end{cases}$$

Thus we may study the structure of Gluskin algebras by making use of their quasi-triangular representation. In this way we obtain easily:

THEOREM 3. *A Gluskin algebra is solvable if and only if all the factors $X_{\alpha\alpha}$ are one-dimensional or, equivalently, if $r = n$.*

THEOREM 4. *A Gluskin algebra is reductive if and only if all the blocks $X_{\alpha\beta} (\alpha \neq \beta)$ are zero matrices, or equivalently, if the graph G S consists of cycles only.*

A clear insight into this subject will be given in the thesis where a larger class of algebras is studied.

References

[1] J. Bello, *Algebras generated by transvection matrices*, doctoral thesis (in preparation).

[2] L. M. Gluskin, *Authomorphisms of multiplicative semigroups of matrix algebras* (in Russian), Usp. Mat. Nauk, vol. 11, No 1 (1956), 199-206.

***) i.e. whose entries are arbitrary.