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Stability in generalized pseudo-dynamical systems [1)]

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We shall give some elementary idea of a certain uniform and relatively general presentation of stability concepts for generalized dynamical systems without uniqueness called, in accordance with [5], generalized pseudo-dynamical systems. In this way, some stability properties of solutions (or of sets of solutions) of ordinary differential equations without uniqueness may be generalized. We shall limit ourselves to only one possibility of a simple case of stability concepts. Some other concepts, corresponding to various notions of stability, semi-stability etc., considered for dynamical systems with uniqueness, may be introduced and investigated in a similar manner; we shall give only one definition in the last section.

Generalized dynamical systems without uniqueness have been considered for instance by I. U. Bronshtain [1—4], E. Roxin [9] and A. Pelczar [6, 8].

1. Let X be a non-empty set, called space in the sequel, $(G, +)$ be an abelian semi-group having the neutral element 0, A be a subset of $G\times X$ such that

(1) $$\{0\}\times X \subset A$$

and finally let λ be a mapping

(2) $$A \ni (t, x) \to \lambda(t, x) \in \mathcal{S}(X)\,,$$

where by $\mathcal{S}(X)$ we denote the collection of all non-empty subsets of the space X.

For $x \in X$ we put

(3) $$I_x \stackrel{\text{df}}{=} \{t \in G\colon\ (t, x) \in A\}\,.$$

Observe that

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1) The main results of this paper had been presented at July, 1, 1975 during the international conference on differential equations and applications held in Ruse (Bulgaria) between June, 30 and July, 4, 1975. A short summary is published as a preprint in proceedings of this conference (the title of proceedings in bulgarish: конференция по диференцални уравнения и приложения, Разюмета, Русе, 30 юни — 4 юли, 1975 година), p. 141.

$0 \in I_x$ for every $x \in X$ and then $I_x \neq \emptyset$ for every $x \in X$.

Definition 1. We say that $(X, G, A; \lambda)$ is a *generalized local pseudo-dynamical system* (shortly: GLPD-system) if and only if

(4) $$\lambda(0, x) = \{x\} \quad \text{for every } x \in X,$$

(5) $$\text{if } t \in I_x, y \in \lambda(t, x), s \in I_y, \quad \text{then } t+s \ (= s+t) \in I_x$$

and

(6) $$\lambda(s, y) \subset \lambda(s+t, x) \quad \text{for every } x \in X, t \in I_x, y \in \lambda(t, x), s \in I_y.$$

If $A = G \times X$ then we say that $(X, G, A; \lambda)$ is a generalized (global) pseudo-dynamical system and we denote it by $(X, G; \lambda)$; such systems are considered in [6].

For any $x \in X$ we put

(7) $$\lambda(x) \overset{\text{df}}{=} \bigcup \{\lambda(t, x): t \in I_x\}.$$

Definition 2. We say that a mapping

$$\beta: X \to \mathcal{S}(\mathcal{S}(X))$$

(where — of course — $\mathcal{S}(\mathcal{S}(X))$ denotes the collection of all non-empty subsets of $\mathcal{S}(X)$) is normal (or shortly $\beta \in \mathcal{N}$) if and only if every point x of the space X belongs to each set B being an element of the family $\beta(x)$.

Remark 1. In [6] there were considered such normal mappings fulfilling some additional conditions. It is easy to see that in each topological space the mapping $x \mapsto$ {the family of neighbourhoods of x} and the mapping $x \mapsto$ {the fundamental basis of neighbourhoods of x} are obviously normal.

2. Let us consider a GLPD-system $(X, G, A; \lambda)$ fixed throughout this paper. Let Ω be a subfamily of $\mathcal{S}(X)$, β be a normal mapping and let M be a non-empty subset of X.

Definition 3. We say that the set M fulfils the condition $S(\Omega, \beta; \lambda)$ (shortly: $M \in S(\Omega, \beta; \lambda)$) or that M is (Ω, β)-*stable with respect to* λ, if and only if

for every $Q \in \Omega$ and every $x \in M$

there exists a set $B \in \beta(x)$

such that

(8) $$y \in B \Rightarrow \lambda(y) \subset Q.$$

Definition 4. If $\mathcal{A}$ and $\mathcal{B}$ are two subfamilies of $\mathcal{S}(X)$, then

$$\mathcal{A} \succ \mathcal{B} \overset{\text{df}}{\Leftrightarrow} \{\text{for every } B \in \mathcal{B} \text{ there is } A \in \mathcal{A} \text{ such that } A \subset B\}.$$

Definition 5. If $A \subset X$ then we say that A is λ-invariant (or shortly: invariant) if and only if $\lambda(x) \subset A$ for every $x \in A$.

Definition 6. We say that the set M fulfils the condition $L(\Omega, \beta; \lambda)$ (shortly: $M \in L(\Omega, \beta; \lambda)$) if and only if there exists a family $\mathcal{A} \subset \mathcal{T}(X)$ such that

(a) $\beta(y) > \mathcal{A}$ for every $y \in M$,

(b) $\mathcal{A} > \Omega$,

(c) every set A belonging to $\mathcal{A}$ is λ-invariant.

THEOREM 1. *For every $\Omega \subset \mathcal{T}(X)$ and every $\beta \in \mathcal{N}$ the conditions $S(\Omega, \beta; \lambda)$ and $L(\Omega, \beta; \lambda)$ are equivalent.*

Proof. 1° Suppose that $M \in L(\Omega, \beta; \lambda)$. Let $y \in M$ and $Q \in \Omega$ be given. There exists a set $A \in \mathcal{A}$ such that $A \subset Q$ (see the condition (b)). For A and y we can choose $B \in \beta(y)$ such that $B \subset A$ (see (a)). Hence, if $z \in B$ then $z \in A$, and then, in virtue of (c), $\lambda(z) \subset A$, which implies that $\lambda(z) \subset Q$. The proof of the implication $L \Rightarrow S$ is finished.

2° Suppose now that $M \in S(\Omega, \beta; \lambda)$. Let us put for any subset D of the space X

$$E(D) \stackrel{\mathrm{df}}{=} \{z \in X:\ \lambda(z) \subset D\}, \tag{9}$$

and

$$\mathcal{E} = \{E(Q):\ Q \in \Omega\}. \tag{10}$$

We shall show that putting $\mathcal{A} = \mathcal{E}$ we obtain a family fulfilling the conditions (a)—(c) of Definition 6.

Let $x \in M$ and $Q \in \Omega$ be given. We can find $B \in \beta(x)$ such that the implication (8) holds true. But (8) is equivalent to saying that $B \subset E(Q)$. This means that (a) is satisfied. Moreover, if $Q \in \Omega$ then $E(Q) \subset Q$; indeed for $y \in E(Q)$ we have $\lambda(y) \subset Q$ and then in particular $\lambda(0, y) = \{y\} \subset Q$. Finally we shall prove (c). Let us consider a set $E(Q)$. Suppose that $y \in E(Q)$. Hence $\lambda(y) \subset Q$. Let $t \in I_y$ be fixed and let us consider $\lambda(t, y)$; it is sufficient to show that $\lambda(t, y) \in E(Q)$, because of the fact that t is fixed arbitrarily. If $z \in \lambda(t, y)$, then for every $s \in I_z$ we have:

$$\lambda(s, z) \subset \lambda(s+t, y) \subset Q .$$

Thus $\lambda(z) \subset Q$, which means that $z \in E(Q)$; then $\lambda(t, y) \in E(Q)$. The proof is finished.

3. Let $(T, \leqslant)$ be a partially ordered space. We put $P = T \setminus \{\inf T\}$ if $\inf T$ exists and $P = T$ if $\inf T$ does not exist. Let W be an invariant subset of X, $W \neq \emptyset$. Let finally M be a non-empty subset of X.

Definition 7. A mapping

$$V: W \rightarrow T$$

is said to be a *Liapunov function of the type $(W, P; \Omega, \beta)$ for the set M* (or shortly: V is Liapunov $(W, P; \Omega, \beta)$-function) if and only if

(I) the family

$$\mathcal{A} = \{A_\eta \colon \eta \in P\}$$

where

(11) $$A_\eta = \{y \in W \colon V(y) \leqslant \eta\}$$

fulfils the conditions (a) and (b) from Definition 6 and moreover

(II) for every $y \in W$ and every $t \in I_y$ the following inequality

(12) $$V(w) \leqslant V(y)$$

holds true for every $w \in \lambda(t, y)$.

THEOREM 2. *If a set $M \subset X$, $M \neq \emptyset$ is such that there exists a Liapunov $(W, P; \Omega, \beta)$-function for M (with some invariant subset W of the space X and some partially ordered space T, where $P = T \setminus \{\inf T\}$ or $P = T\}$) then M fulfils the condition $S(\Omega, \beta; \lambda)$.*

Proof. Let V be a Liapunov $(W, P; \Omega, \beta)$-function for M. The conditions (a) and (b) of Definition 6 are satisfied because of the fact that V is the Liapunov function. We shall show below that also the condition (c) is satisfied for the family $\mathcal{A}$ defined by (11). Let z be an arbitrary point of A_η. For every $t \in I_z$ and every $w \in \lambda(t, z)$ we have:

$$V(w) \leqslant V(z) \leqslant \eta$$

Hence $w \in A_\eta$. Then $\lambda(t, z) \subset A_\eta$ for every $t \in I_z$ and consequently $\lambda(z) \subset A_\eta$. This means that A_η is invariant. Then the condition $L(\Omega, \beta; \lambda)$ is satisfied; in virtue of Theorem 1 this means that $M \in (\Omega, \beta; \lambda)$.

4. Assume now that the partially ordered space T is such that for every $S \subset T$, $S \neq \emptyset$ there exists $\inf S$ and put — as in the previous section — $P = T \setminus \{\inf T\}$. Furthermore we assume the following:

(A) for each $\lambda \in P$, there exists $\mu \in P$ such that $\mu < \lambda$; for every non-empty subset S of T and for every $\varrho \in P$ such that $\inf S < \varrho$, there exists $\eta \in S$ such that $\eta \leqslant \varrho$.

Suppose that $\Omega = \{Q_\eta\}_{\eta \in P}$ and

(B) if $\eta, \lambda \in P$ and $\eta \leqslant \lambda$, then $Q_\eta \subset Q_\lambda$.

Let β be a normal mapping and let M be a non-empty subset of X.

THEOREM 3. *Under the above assumptions, if M is (Ω, β)-stable with respect to λ, then there exists an invariant set W such that for every $x \in M$ there exists $B \in \beta(x)$ contained in W, and there exists a Liapunov $(W, P; \Omega, \beta)$-function for M.*

Proof. Let us consider $\sigma \in P$, arbitrarily fixed. Let us put

(13) $$W = \{x \in X \colon \lambda(x) \subset Q_\sigma\}.$$

We shall show that W fulfils all conditions required in the assertion of the theorem. Let $x \in M$ be fixed. Since $M \in S(\Omega, \beta; \lambda)$, there is a set $B \in \beta(x)$ such that

$$y \in B \Rightarrow \lambda(y) \subset Q_\sigma;$$

this means, however, that $B \subset W$. Observe here that in particular $M \subset W$.

If $x \in W$ and $t \in I_x$, then for every $w \in \lambda(t, x)$ we have $\lambda(s, w) \in \lambda(x)$ for $s \in I_w$. This means that for every $w \in \lambda(t, x)$ it is: $\lambda(w) \subset \lambda(x) \subset Q_\sigma$; thus $w \in W$. But this means finally that $\lambda(x) \subset W$ and then the proof of the invariance of W is finished.

Now we shall define the Liapunov function. Let us put for $x \in W$:

$$V(x) \overset{\text{df}}{=} \inf\{\eta \in P\colon\ \lambda(x) \subset Q_\eta\} \tag{14}$$

and (see (11))

$$A_\eta = \{y \in W\colon\ V(y) \leqslant \eta\}\,.$$

We shall show that the family $\mathcal{A} = \{A_\eta\}$ fulfils the conditions (a) and (b) of Definition 6.

Let $Q_\eta \in \Omega$ be fixed. Consider $\eta^0 \in P$ such that $\eta^0 < \eta$. We shall show that $A_{\eta^0} \subset Q_\eta$. If y A_{η^0} then

$$V(y) = \inf\{\mu \in P\colon\ \lambda(y) \subset Q_\mu\} \leqslant \eta^0 < \eta\,.$$

We can choose (see (A)) $\sigma \in P$ such that $\sigma \leqslant \eta$ and $\lambda(y) \subset Q_\sigma$. This, however, gives obviously:

$$y \in \lambda(y) \subset Q_\sigma \subset Q_\eta$$

and then $y \in Q_\eta$. The proof of the condition (b) is finished. Let $y \in M$ and A_η be fixed. Let us consider $\mu = \inf\{\eta, \sigma\}$ and the set A_μ (obviously contained in A_η). Note that from the condition (A) it follows easily that $\mu \in P$. For μ and y we can choose $B \in \beta(y)$ such that

$$\lambda(x) \subset Q_\mu \quad \text{for } x \in B\,. \tag{15}$$

Since $Q_\mu \subset Q_\sigma$, the condition (15) means that $B \subset W$ and then the values $V(x)$ are well defined for $x \in B$. Moreover we have

$$V(x) \leqslant \mu \leqslant \eta \quad \text{for } x \in B\,. \tag{16}$$

The above condition (16) implies that $B \subset A_\mu \subset A_\eta$. In this way condition (a) has been verified. In order to prove (12) we consider $y \in W$, $t \in I_y$ and $w \in \lambda(t, y)$; we have

$$V(w) = \inf\{\eta \in P\colon\ \lambda(w) \subset Q_\eta\} \leqslant \inf\{\eta \in P\colon\ \lambda(y) \subset Q_\eta\} = V(y)\,.$$

The proof is finished.

Remark 2. Condition (A) is obviously satisfied for $T = [0, \infty)$. Moreover, this condition is also satisfied for many ordered spaces. There are, however, some partially ordered spaces (but not ordered) for which this condition is also satisfied. The author hopes that some examples will be given with certain applications separately.

5. The concept of (Ω, β)-stability may be generalized and modified in various manners. In particular, the following definition may be stated:

Definition 8. A set M is said to be semi-(Ω, β)-*stable with respect to* λ if and only if for each $x \in M$ and each $Q \in \Omega$, there exists $u \in I_x$ and there exists $B \in \beta(x)$ such that $\lambda(y) \subset Q$ for every $y \in \lambda(u, z)$ and every $z \in B$ such that $u \in I_z$. This is a direct generalization of the concept of semi-stability considered in [7] with respect to local pseudo-dynamical systems. Many other possibilities of such generalizations may be inspired by the notions investigated by — for instance — P. Habets and K. Peiffer in [5].

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