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CCCCLXXV PRACE MATEMATYCZNE, ZESZYT 19 1977

A continuous and a discrete boundary value problem

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0. Introduction. The purpose of this paper is to study the question of the existence and the uniqueness of solutions, respectively, of the continuous boundary value problem

$$x'' = f(t, x, x')\,, \tag{0.1}$$

$$x(0) = 0\,, \quad x(1) = 0\,, \tag{0.2}$$

where $f\colon X = [0, 1]\times R^d\times R^d\to R^d$, and of the discrete boundary value problem

$$x_i = g(i, x_i, \Delta x_i)\,, \quad i = 1, \ldots, m-1\,, \tag{0.3}$$

$$x_0 = 0\,, \quad x_m = 0\,, \tag{0.4}$$

where $g\colon Y_m = \{0, 1, \ldots, m\}\times R^d\times R^d\to R^d$, and to give a theorem such that, as the mesh is refined to zero, the solutions of the appropriately defined discrete problems of the form (0.3), (0.4) converge to the solution of problem (0.1), (0.2).

In the sequel we shall assume that the function f satisfies the Carathéodory condition (i.e. 1° for every $u, v\in R^d$, $f(\cdot, u, v)$ is measurable on $[0, 1]$, 2° for almost every $t\in[0, 1]$, $f(t, \cdot, \cdot)$ is continuous on $R^d\times R^d$) and there exist functions $p_i\in L^+_{s_i}$ (see Notations) $i = 0, 1, 2$, $s_0 = 1$, $s_1\geqslant 1$, $s_2\geqslant 2$ satisfying the condition

$$c = \pi^{-2}\left(\frac{\pi^2}{4}\right)^{\frac{1}{s_1}}|p_1|_{s_1}+\frac{1}{4}(4)^{\frac{1}{s_2}}|p_2|_{s_2}<1 \tag{0.5}$$

such that

$$|f(t, u, v)|\leqslant p_0(t)+p_1(t)|u|+p_2(t)|v| \tag{0.6}$$

for $u, v\in R^d$, $t\in[0, 1]$. In particular, the last assumption is fulfilled if the function f satisfies the Lipschitz condition

$$|f(t, u, v)-f(t, \bar u, \bar v)|\leqslant p_1(t)|u-\bar u|+p_2(t)|v-\bar v|\,, \tag{0.7}$$

where $p_i\in L^+_{s_i}$, $i = 1, 2$, $s_1\geqslant 1$, $s_2\geqslant 2$ and $f(\cdot, 0, 0)\in L_1$.

As regards the function g, we shall assume that it is continuous and satisfies the condition

$$(0.8)\qquad |g(i,u,v)| \leqslant q_i + \bar{q}_i|u| + \bar{\bar{q}}_i|v|, \qquad i = 1, \dots, m-1,$$

where the vectors $q = (q_0, \dots, q_m)$, $\bar{q} = (\bar{q}_0, \dots, \bar{q}_m)$, $\bar{\bar{q}} = (\bar{\bar{q}}_0, \dots, \bar{\bar{q}}_m)$ have non-negative coordinates and obey the inequality

$$(0.9)\qquad c_m = \lambda_m^2 \left(\frac{m}{4\lambda_m^2}\right)^{\frac{1}{s_1}} |\bar{q}|_{s_1} + \mu_m^2 \left(\frac{m}{4\mu_m^4}\right)^{\frac{1}{s_2}} |\bar{\bar{q}}|_{s_2} < 1,$$

where $\lambda_m = \left(2\sin\frac{\pi}{2m}\right)^{-1}$, $\mu_m^2 = \frac{1}{2}\left[\frac{m+1}{2}\right]$, ($[x]$ denotes the whole part of x). It is apparent that the Lipschitz condition

$$(0.10)\qquad |g(i,u,v) - g(i,\bar{u},\bar{v})| \leqslant \bar{q}_i|u-\bar{u}| + \bar{\bar{q}}_i|v-\bar{v}|,$$

$i = 1, \dots, m-1$, $u, \bar{u}, v, \bar{v} \in R^d$, implies the relation (0.8).

The proofs of the existence theorems are based on the theory of the topological degree of completely continuous mapping, more precisely we apply the Leray-Schauder Alternative [4]. For a priori estimates we follow the concepts of M. P. Colautti [2] and Z. Opial [8] using inequalities of Wirtinger's and Opial's types.

Similar results for one equation with the right-hand side satisfying more restrictive assumptions were given by A. Lasota [7].

Section 2 of this paper contains some generalizations of Wirtinger's and Opial's inequalities in continuous and discrete cases. In Section 3 a priori estimates for some differential and difference inequalities are given. The existence and the uniqueness theorems are contained in Section 4. The next section deals with the approximation theorem.

1. Notations. We shall denote the Euclidean norm and scalar product in d-dimensional space R^d by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. If $b \in (R^d)^{m+1}$, $b = \{b_i\}$, $b_i \in R^d$, then by definition

$$\|b\| = \max\{|b_i|:\ i = 0, \dots, m\}.$$

For $a \in R^d$, $a = (a_1, \dots, a_d)$, $s \geqslant 1$ we shall put $|a|_s = \left(\sum_{j=1}^{d} |a_j|^s\right)^{\frac{1}{s}}$. Let L_s be the linear space of all real functions defined and summable with p-th powers on the interval $[0,1]$, endowed with its usual norm

$$|p|_s = \left(\int_0^1 |p(t)|^s dt\right)^{\frac{1}{s}}.$$

The subset $L_s^+ \subset L_s$ contains all non-negative functions. C^d will denote the linear space of all real d-dimensional vector functions defined and continuous on the interval $[0,1]$, endowed with the norm: $\|x\| = |u|_2$, where $u(t) = |x(t)|$, $t \in [0,1]$, $x \in C^d$. The set $C_1^d \subset C^d$ consists of all continuously differentiable functions. Let $x = (x_0, \dots, x_m) \in (R^d)^{m+1}$. The symbols Δ and ∇ denote the difference operators from $(R^d)^{m+1}$ to $(R^d)^{m+1}$, as follows: $\Delta x = (\Delta x_0, \dots, \Delta x_{m-1}, 0)$, $\nabla x = (0, \nabla x_1, \dots, \nabla x_m)$, where

$$\Delta x_i = \begin{cases} x_{i+1} - x_i, & i = 0, \dots, m-1, \\ 0, & i = m, \end{cases}$$

$$\nabla x_i = \begin{cases} 0, & i = 0, \\ x_i - x_{i-1}, & i = 1, \dots, m. \end{cases}$$

2. Continuous and discrete inequalities of Wirtinger's and Opial's types. We shall start by recalling Wirtinger's inequality.

LEMMA 2.1. *If $x \in C_1^d$ and $x(0) = x(1) = 0$, then*

$$\int_0^1 |x(t)|^2 dt \leqslant \frac{1}{\pi^2} \int_0^1 |x'(t)|^2 dt. \tag{2.1}$$

The value π^{-2} is the smallest possible constant.

The proof for the case $d = 1$ can be found, e.g. in [5]. The proof for the vector function follows directly from applying (2.1) to every coordinate.

The following inequality is true:

LEMMA 2.2. [10] *If $p \in L_1^+$, $x \in C_1^d$ and $x(0) = x(1) = 0$, then*

$$\int_0^1 p(t)|x(t)|^2 dt \leqslant \frac{1}{4} \int_0^1 p(t)\, dt \int_0^1 |x'(t)|^2 dt. \tag{2.2}$$

The value $\frac{1}{4}$ is the smallest possible constant.

The next lemma is a generalization of (2.1) and (2.2).

LEMMA 2.3. *Let $p \in L_s^+$, $s \geqslant 1$. If $x \in C_1^d$ and $x(0) = x(1) = 0$, then*

$$\int_0^1 p(t)|x(t)|^2 dt \leqslant \frac{1}{\pi^2} \left(\frac{\pi^2}{4}\right)^{\frac{1}{s}} |p|_s \int_0^1 |x'(t)|^2 dt. \tag{2.3}$$

Proof. The case $s = 1$ is contained in Lemma 2.2. Let $s > 1$ and $s' \in R$ be such that $\frac{1}{s} + \frac{1}{s'} = 1$. From Hölder's inequality and Lemmas 2.1 and 2.2 the following relations may be obtained:

$$\int_0^1 p(t)|x(t)|^2 dt \leqslant \left(\int_0^1 p^s(t)|x(t)|^2 dt\right)^{\frac{1}{s}} \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{s'}} \leqslant$$

$$\leqslant \left(\frac{1}{4}\right)^{\frac{1}{s}} |p|_s \left(\int_0^1 |x'(t)|^2 dt\right)^{\frac{1}{s}} \left(\frac{1}{\pi^2}\right)^{\frac{1}{s'}} \left(\int_0^1 |x'(t)|^2 dt\right)^{\frac{1}{s'}}.$$

Remark 2.1. If we put $p(t) = 1$ on all $[0,1]$ and if we pass to the limit in (2.3) as $s \to \infty$, then we obtain the inequality (2.1).

Z. Opial proved the following inequality:

LEMMA 2.4. [9] *If* $x \in C_1^d$ *and* $x(0) = x(1) = 0$, *then*

$$\int_0^1 |x(t)|\,|x'(t)|\,dt \leqslant \frac{1}{4}\int_0^1 |x'(t)|^2\,dt . \tag{2.4}$$

The value $\frac{1}{4}$ *is the smallest possible constant.*

Now we have an inequality of a similar type to that in Lemma 2.2.

LEMMA 2.5. [10] *Let* $p \in L_2^+$. *If* $x \in C_1^d$ *and* $x(0) = x(1) = 0$, *then*

$$\int_0^1 p(t)\,|x(t)|\,|x'(t)|\,dt \leqslant \frac{1}{2}\Big(\int_0^1 p^2(t)\,dt\Big)^{\frac{1}{2}}\int_0^1 |x'(t)|^2\,dt . \tag{2.5}$$

The proof follows immediately from Schwarz's inequality and Lemma 2.2.

The next inequality is a generalization of those from (2.4) and (2.5).

LEMMA 2.6. *Let* $p \in L_s^+$, $s \geqslant 2$. *If* $x \in C_1^d$ *and* $x(0) = x(1) = 0$, *then*

$$\int_0^1 p(t)\,|x(t)|\,|x'(t)|\,dt \leqslant \frac{1}{4}\,(4)^{\frac{1}{s}}\,|p|_s\int_0^1 |x'(t)|^2\,dt . \tag{2.6}$$

Proof. Lemma 2.5 includes the case $s = 2$. Let $s > 2$ and $\frac{2}{s} + \frac{1}{s'} = 1$. By Hölder's inequality and Lemmas 2.4 and 2.5 we have

$$\int_0^1 p(t)\,|x(t)|\,|x'(t)|\,dt \leqslant \Big(\int_0^1 p^{\frac{s}{2}}(t)\,|x(t)|\,|x'(t)|\,dt\Big)^{\frac{2}{s}}\Big(\int_0^1 |x(t)|\,|x'(t)|\,dt\Big)^{\frac{1}{s'}}$$

$$\leqslant (\tfrac{1}{4})^{\frac{1}{s}}|p|_s\Big(\int_0^1 |x'(t)|^2\,dt\Big)^{\frac{2}{s}}(\tfrac{1}{4})^{\frac{1}{s'}}\Big(\int_0^1 |x'(t)|^2\,dt\Big)^{\frac{1}{s'}} .$$

Remark 2.2 If we put $p(t) = 1$ for $t \in [0,1]$ and pass to the limit in (2.6) as $s \to \infty$, then we obtain the inequality (2.4).

The next few lemmas will be the discrete analogues of the previous ones. The discrete case of Wirtinger's inequality which follows is due to Ky Fan, O. Taussky and J. Todd [6] (see also [7]).

LEMMA 2.7. *If a vector* $u = (u_0, \ldots, u_m) \in (R^d)^{m+1}$, $u_i \in R^d$, $i = 0, \ldots, m$, *satisfies the boundary condition* $u_0 = u_m = 0$, *then*

$$\sum_{i=0}^{m} |u_i|^2 \leqslant \lambda_m^2 \sum_{i=0}^{m-1} |\Delta u_i|^2 , \tag{2.7}$$

where $\lambda_m = \Big(2\sin\dfrac{\pi}{2m}\Big)^{-1}$. λ_m *is the best possible coefficient.*

The proof for the case $d > 1$ is a direct consequence of the case for $d = 1$.

LEMMA 2.8. *Let* $p = (p_0, \ldots, p_m) \in R^{m+1}$, $p_i \geqslant 0$, $i = 0, \ldots, m$. *If a vector* $u = (u_0, \ldots, u_m) \in (R^d)^{m+1}$ *satisfies condition* $u_0 = u_m = 0$, *then*

$$\sum_{i=0}^{m} p_i |u_i|^2 \leqslant \frac{m}{4} \sum_{i=0}^{m} p_i \sum_{j=0}^{m-1} |\Delta u_j|^2 . \tag{2.8}$$

For even m *the value* $\frac{m}{4}$ *is the smallest constant.*

Proof. From the conditions $u_0 = u_m = 0$ and the definition of the operator Δ, we have

$$u_i = \sum_{j=0}^{i=1} \Delta u_j, \qquad u_i = -\sum_{j=i}^{m-1} \Delta u_j, \qquad i = 1, \ldots, m-1 .$$

Hence

$$|u_i| \leqslant \frac{1}{2} \sum_{j=0}^{m-1} |\Delta u_j|, \qquad i = 1, \ldots, m-1 ,$$

and, using Cauchy's inequality, we obtain

$$\sum_{i=1}^{m} p_i |u_i|^2 \leqslant \frac{1}{4} \sum_{i=0}^{m} p_i \Big(\sum_{j=0}^{m-1} |\Delta u_j| \Big)^2 \leqslant \frac{m}{4} \sum_{i=0}^{m} p_i \sum_{j=0}^{m-1} |\Delta u_j|^2 .$$

The vectors $u_i^j = \frac{1}{2} m - \left| i - \frac{1}{2} m \right|$, $i = 0, \ldots, m$, $j = 1, \ldots, d$, $p_i = 0$ for $i \neq \frac{m}{2}$, $p_i = 1$ for $i = \frac{m}{2}$ give the equality in (2.8) for even m.

The next lemma gives the generalization of inequalities (2.7) and (2.8).

LEMMA 2.9. *Let* $s \geqslant 1$. *According to the assumptions of Lemma* 2.8, *the following inequality holds*:

$$\sum_{i=0}^{m} p_i |u_i|^2 \leqslant \lambda_m^2 \left(\frac{m}{4\lambda_m^2} \right)^{\frac{1}{s}} |p|_s \sum_{i=0}^{m-1} |\Delta u_i|^2 . \tag*{2.9)}$$

Proof. The case $s = 1$ has been dealt with in Lemma 2.8. Let $s > 1$ and $\frac{1}{s} + \frac{1}{s'} = 1$. From Hölder's inequality and Lemmas 2.7 and 2.8, we have

$$\begin{aligned} \sum_{i=0}^{m} p_i |u_i|^2 &\leqslant \Big(\sum_{i=0}^{m} p_i^s |u_i|^2 \Big)^{\frac{1}{s}} \Big(\sum_{i=0}^{m} |u_i|^2 \Big)^{\frac{1}{s'}} \\ &\leqslant \left(\frac{m}{4} \right)^{\frac{1}{s}} \Big(\sum_{i=0}^{m} p_i^s \Big)^{\frac{1}{s}} \Big(\sum_{i=0}^{m-1} |\Delta u_i|^2 \Big)^{\frac{1}{s}} (\lambda_m^2)^{\frac{1}{s'}} \Big(\sum_{i=0}^{m-1} |\Delta u_i|^2 \Big)^{\frac{1}{s'}} \end{aligned}$$

A simple calculation concludes the proof.

Remark 2.3. If in Lemma 2.9, $p_i = 1$ for $i = 0, \dots, m$, then (2.9) gives inequality (2.7) as $s \to \infty$.

The next inequality, introduced for the case $d = 1$ by A. Lasota, is the discrete equivalent of Z. Opial's inequality.

LEMMA 2.10. [7] *If vector* $u = (u_0, \dots, u_m) \in (R^d)^{m+1}$ *fulfils the condition* $u_0 = u_m = 0$, *then*

$$\sum_{i=1}^{m-1} |u_i|\,|\Delta u_i| \leqslant \mu_m^2 \sum_{i=0}^{m-1} |\Delta u_i|^2 , \tag{2.10}$$

where $\mu_m^2 = \frac{1}{2}\left[\frac{m+1}{2}\right]$ *(*$[x]$ *denotes the whole part of* x*). For even* m, μ_m^2 *is the smallest possible constant.*

The proof of (2.10) for the case $d > 1$ can be performed analogously as in the case of $d = 1$ (see [7]).

Now we have the inequality:

LEMMA 2.11. *Let* $p = (p_0, \dots, p_m) \in R^{m+1}$, $p_i \geqslant 0$, $i = 0, \dots, m$. *If a vector* $u = (u_0, \dots, u_m) \in (R^d)^{m+1}$ *fulfils the condition* $u_0 = u_m = 0$, *then*

$$\sum_{i=0}^{m-1} p_i |u_i|\,|\Delta u_i| \leqslant \left(\frac{m}{4}\right)^{\frac{1}{2}} \Big(\sum_{i=0}^{m} p_i^2\Big)^{\frac{1}{2}} \sum_{i=0}^{m-1} |\Delta u_i|^2 . \tag{2.11}$$

Proof. By Cauchy's inequality and Lemma 2.8, we have

$$\sum_{i=1}^{m-1} p_i |u_i|\,|\Delta u_i| \leqslant \Big(\sum_{i=0}^{m} p_i^2 |u_i|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=0}^{m-1} |\Delta u_i|^2\Big)^{\frac{1}{2}}$$

$$\leqslant \left(\frac{m}{4}\right)^{\frac{1}{2}} \Big(\sum_{i=0}^{m} p_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i=0}^{m-1} |\Delta u_i|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=0}^{m-1} |\Delta u_i|^2\Big)^{\frac{1}{2}} .$$

We shall now *give a generalization of inequalities (2.10) and (2.11).*

LEMMA 2.12. *Let* $s \geqslant 2$. *From the assumptions of Lemma* 2.11 *the following inequality holds:*

$$\sum_{i=0}^{m-1} p_i |u_i|\,|\Delta u_i| \leqslant \mu_m^2 \left(\frac{m}{4\mu_m^4}\right)^{\frac{1}{s}} |p|_s \sum_{i=0}^{m-1} |\Delta u_i|^2 , \tag{2.12}$$

where $\mu_m^2 = \frac{1}{2}\left[\frac{m+1}{2}\right]$.

Proof. The case $s = 2$ was treated in Lemma 2.11. Let $s > 2$ and $\frac{2}{s} + \frac{1}{s'} = 1$.

On the base of Hölder's inequality and Lemmas 2.10 and 2.11, we obtain:

$$\sum_{i=0}^{m-1} p_i|u_i|\,|\Delta u_i| \leqslant \Big(\sum_{i=0}^{m-1} p_i^{\frac{s}{2}}|u_i|\,|\Delta u_i|\Big)^{\frac{2}{s}}\Big(\sum_{i=0}^{m-1} |u_i|\,|\Delta u_i|\Big)^{\frac{1}{s'}}$$

$$\leqslant \Big(\frac{m}{4}\Big)^{\frac{1}{s}}|p|_s\Big(\sum_{i=0}^{m-1} |\Delta u_i|^2\Big)^{\frac{2}{s}}(\mu_m^2)^{\frac{1}{s'}}\Big(\sum_{i=0}^{m-1} |\Delta u_i|^2\Big)^{\frac{1}{s'}}.$$

Remark 2.4. If $p_i = 1$ for $i = 0, \ldots, m$, and in inequality (2.12) we proceed to the limit as $s \to \infty$, we obtain (2.10).

3. A priori estimates. Using the results of the previous section, we shall prove two lemmas giving an estimate of the solutions of certain differential and difference inequalities.

LEMMA 3.1. *Let* $p_i \in L_{s_i}^{+}$, $i = 0, 1, 2$, *be functions satisfying the condition*

$$c = \frac{1}{\pi^2}\Big(\frac{\pi^2}{4}\Big)^{\frac{1}{s_1}}|p_1|_{s_1} + \frac{1}{4}(4)^{\frac{1}{s_2}}|p_2|_{s_2} < 1, \tag{3.1}$$

for certain $s_0 = 1$, $s_1 \geqslant 1$, $s_2 \geqslant 2$. *If a function* $x \in C^d$ *has an absolutely continuous derivative* x *and fulfils the boundary condition*

$$x(0) = 0, \qquad x(1) = 0 \tag{3.2}$$

and inequality

$$|x''(t)| \leqslant p_0(t) + p_1(t)|x(t)| + p_2(t)|x'(t)| \tag{3.3}$$

almost everywhere in [0,1], *then*

$$\|x\| \leqslant \frac{|p_0|_1}{2\pi(1-c)}, \quad \|x'\| \leqslant \frac{|p_0|_1}{2(1-c)}. \tag{3.4}$$

Proof. Multiplying both sides of (3.3) by $|x(t)|$ and integrating over [0,1], we obtain

$$\int_0^1 |x''(t)|\,|x(t)|\,dt \leqslant \int_0^1 p_0(t)|x(t)|\,dt + \int_0^1 p_1(t)|x(t)|^2 dt + \int_0^1 p_2(t)|x(t)|\,|x'(t)|\,dt \tag{3.5}$$

From the formula for integration by parts and assumption (3.2) and also Schwarz's inequality for the scalar product, we have the relations

$$\int_0^1 |x'(t)|^2 dt = \int_0^1 |\langle x''(t), x(t)\rangle|\,dt \leqslant \int_0^1 |x''(t)|\,|x(t)|\,dt. \tag{3.6}$$

Condition (3.2) implies

$$|x(t)| \leqslant \frac{1}{2}\int_0^1 |x'(t)|\,dt \qquad \text{for } t \in [0,1].$$

Thus, on the basis of Schwarz's inequality

$$(3.7)\qquad \int_0^1 p_0(t)\,|x(t)|\,dt \leqslant \frac{1}{2}\int_0^1 p_0(t)\,dt \int_0^1 |x'(t)|\,dt = \frac{1}{2}\,|p_0|_1 \Big(\int_0^1 |x'(t)|^2\,dt\Big)^{\frac{1}{2}} .$$

Applying the deduced relations (3.6) and (3.7) and also Lemmas 2.3 and 2.6 to the inequality (3.5), we obtain:

$$\|x'\|^2 \leqslant \frac{1}{2}\,|p_0|_1\|x'\| + \frac{1}{\pi^2}\left(\frac{\pi^2}{4}\right)^{\frac{1}{s_1}} |p_1|_{s_1}\|x'\|^2 + \frac{1}{4}\,(4)^{\frac{1}{s_2}}|p_2|_{s_2}\|x'\|^2 .$$

Hence, for $\|x'\| \neq 0$ we obtain

$$\|x'\|(1-c) \leqslant \frac{1}{2}\,|p_0|_1 ,$$

and from assumption (3.1) we have the second inequality in (3.4). The first is obtained from the second and from Lemma 2.1. The case $\|x'\| = 0$ together with (3.2) implies $\|x\| = 0$.

The difference equivalent of Lemma 3.1 is the following:

LEMMA 3.2. *Let vectors* $q = (q_0, \ldots, q_m)$, $\bar{q} = (\bar{q}_0, \ldots, \bar{q}_m)$, $\bar{\bar{q}} = (\bar{\bar{q}}_0, \ldots, \bar{\bar{q}}_m) \in R^{m+1}$, $q_i, \bar{q}_i, \bar{\bar{q}}_i \geqslant 0$ *satisfy the condition* (0.9) *for certain* $s_1 \geqslant 1$ *and* $s_2 \geqslant 2$. *If a vector* $u = (u_0, \ldots, u_m) \in (R^d)^{m+1}$ *satisfies the boundary condition*

$$(3.8)\qquad u_0 = 0, \qquad u_m = 0$$

and the inequality

$$(3.9)\qquad |\nabla\Delta u_i| \leqslant q_i + \bar{q}_i\,|u_i| + \bar{\bar{q}}_i\,|\Delta u_i| , \qquad i = 1, \ldots, m-1 ,$$

then

$$(3.10)\qquad |u| \leqslant \frac{\sqrt{m\lambda_m}\,|q|_1}{2(1-c_m)} , \qquad |\Delta u| \leqslant \frac{\sqrt{m}\,|q|_1}{2(1-c_m)} .$$

($|u|$ *is the Euclidean norm of* u *in* $(R^d)^{m+1}$).

Proof. As in the proof of the previous Lemma, we shall multiply both sides of inequality (3.9) by $|u_i|$ and sum over $i = 1, \ldots, m-1$. We obtain

$$(3.11)\qquad \sum_{i=1}^{m-1} |\nabla\Delta u_i|\,|u_i| \leqslant \sum_{i=1}^{m-1} q_i\,|u_i| + \sum_{i=1}^{m-1} \bar{q}_i\,|u_i|^2 + \sum_{i=1}^{m-1} \bar{\bar{q}}_i\,|\Delta u_i|\,|u_i| .$$

By assumption (3.8)

$$(3.12)\qquad \sum_{i=0}^{m-1} |\Delta u_i|^2 = \Big|\sum_{i=1}^{m-1} \langle \nabla\Delta u_i, u_i\rangle\Big| \leqslant \sum_{i=1}^{m-1} |\nabla\Delta u_i|\,|u_i| .$$

Also from assumption (3.8) we have the estimate

$$|u_i| \leqslant \frac{1}{2} \sum_{j=0}^{m-1} |\Delta u_j|, \qquad i = 1, \ldots, m-1,$$

from which, on the basis of Cauchy's inequality, we have the relations

$$\sum_{i=1}^{m-1} q_i |u_i| \leqslant \frac{1}{2} \sum_{i=1}^{m-1} q_i \sum_{j=0}^{m-1} |\Delta u_j| \leqslant \frac{\sqrt{m}}{2} |q|_1 \Big(\sum_{j=0}^{m-1} |\Delta u_j|^2 \Big)^{\frac{1}{2}}. \tag{3.13}$$

Applying (3.12), (3.13) and Lemmas 2.9 and 2.12 to (3.11), we obtain

$$|\Delta u|^2 \leqslant \frac{\sqrt{m}}{2} |q|_1 |\Delta u| + \lambda_m^2 \left(\frac{m}{4\lambda_m^2} \right)^{\frac{1}{s_1}} |\bar{q}|_{s_1} |\Delta u|^2 + \mu_m^2 \left(\frac{m}{4\mu_m^4} \right)^{\frac{1}{s_2}} |\bar{\bar{q}}|_{s_2} |\Delta u|^2 .$$

Let $|\Delta u| \neq 0$. Dividing by this expression and bearing in mind the definition of c_m, we get $|\Delta u|(1-c_m) \leqslant \frac{\sqrt{m}}{2} |q|_1$. Hence the assumption (0.9) gives the second relation in (3.10). The first is obtained from the second and Lemma 5.7. The case $|\Delta u| = 0$, together with (3.8) implies $|u| = 0$.

4. Existence and uniqueness theorems. In this sections we shall prove the existence and uniqueness theorems of the solutions to the boundary problem in the differential case (0.1), (0.2) and subsequently in the difference case (0.3), (0.4). Our tool for the proof will be the Leray-Schauder Alternative. The results of the previous section will be used for a priori estimates of solutions to the appropriate equations.

THEOREM 4.1. *If the mapping* $f\colon X \to R^d$ *satisfies Carathéodory's conditions on the set* X *and there exist functions* $p_i \in L_{s_i}^+$, $i = 0, 1, 2$, $s_0 = 1$, $s_1 \geqslant 1$, $s_2 \geqslant 2$ *satisfying condition* (0.5), *such that the inequality* (0.6) *holds, then the boundary problem* (0.1), (0.2) *has at least one solution.*

Proof. Let E be the linear space of all functions defined and having an absolutely continuous derivative on the interval [0,1] and assuming values in R^d. By the norm of $x \in E$ we shall understand $\|x\| + \|x'\|$.

Since the unique solution of the equation $x''(t) = 0$, $t \in [0,1]$, fulfilling the boundary condition (0.2) is the function $x(t) = 0$, $t \in [0,1]$, then, as is well-known from the general theory of differential equations [3], there exists a function $G\colon [0,1]+[0,1] \to R^{d\times d}$, such that the differential boundary problem (0.1), (0.2) is equivalent to the integral problem

$$x(t) = \lambda \int_0^1 G(t,s) f\big(s, x(s), x'(s)\big) ds, \qquad t \in [0.1], \tag{4.1}$$

where $\lambda = 1$. We shall now show that all solutions of (4.1) with any $\lambda \in [0,1]$ are bounded by a common constant. If a function x fulfils equation (4.1), then from