

# Complete classification of the family of all finite sequences of points in a projective space

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## Introduction

Geometry in the sense of F. Klein can be seen as the theory of  $K$ -spaces [5]. The basic problems of this theory concern classification of figures. The classification is called complete if the classes are orbits. Any two elements in an orbit are called congruent.

In Euclidean space, for instance, Frenet's Theorem for curves, Bonnet's Theorem for surfaces, theorems on congruency of triangles and so on, are examples of such classification problems.

In particular, the so-called "discrete figures", considered as finite sets of points, are of great significance. Local properties of such figures as curves or surfaces are usually expressed in terms of properties of some discrete figures, attached to their points. For this reason, it seems to be of interest to study families of discrete figures, especially from the point of view of complete classification.

Two discrete figures are congruent if under some numerations of their points the obtained finite sequences of points are congruent. Thus the classification of discrete figures can be reduced to that of finite sequences of points.

In this paper we give a method for complete projective classification of the family of all discrete figures in a projective space. Roughly speaking, the concept of the cross-ratio for 4-tuples is generalized for arbitrary  $n$ -tuples.

The complete projective classification of the family  $\text{seq } \Pi_*$  of all finite sequences of points in a projective space  $\Pi_*$  is made by using certain complete projective scalar invariants in the way suggested in basic Theorem 1(3.1).

From the fact of possessing one such invariant it immediately follows that a generalization of the result by J. Aczél and others [2], concerning projective scalar invariants of the family of all 4-tuples of points on a projective line, is obtained; see also the papers on the cross-ratio by W. Benz [3], J. A. Thas [8] and S. Topa [10].

Let us note that we are not only interested in obtaining some criteria for projective congruency of elements in  $\text{seq } \Pi_*$ , but also in giving a method of construction of orbits and global sections of the space of all orbits (of the projective group). We shall do this by finding some parametric representations. These results may be used to determine invariants for  $\text{seq } \Pi_*$  not necessarily being of scalar type, for which, for example, may be used the methods presented in [9] [11] as well as the methods of the theory of geometric objects [1] [6] where invariants are called concomitants; see the papers by S. Gołab [4], M. Kucharzewski and S. Węgrzynowski [7], A. Zajtz [12].

The following convention is assumed for referring to formulae (lemmas, propositions, ...)

3(2.4) refers to the 4th formula in the 2nd paragraph in the 3rd chapter

(5.7) — to the 7th formula in 5th paragraph in the current chapter

2(4) — to the 4th paragraph of the 2th chapter.

There are six chapters in the paper, the two of which are in the Appendix.

In Chapter I a survey of notions and facts in the theory of  $g$ -spaces ( $G$ -objects,  $G$ -sets, transformation groups) is given and it is explained what is meant by a  $K$ -space (Klein's space) and by the complete classification of  $g$ -spaces.

It is shown that the classification of the  $K$ -space ( $\text{seq } \Pi_*$ ,  $\Gamma$ ),  $\Gamma$  denoting the projective group for the projective space  $\Pi_*$  over a given vector space  $V$ , may be

reduced to that of  $(\text{seq } V_*, G \times T)$ , where  $G$  is the full linear group for  $V$  and  $T$  the multiplicative group  $F_*^N$  of all  $F_*$ -valued functions on  $N$ ,  $V$  being considered over an arbitrary field  $F$ , the multiplicative group of which is denoted by  $F_*$ ,  $V_* = V - \{0\}$ , and  $N$  denoting the set of all natural numbers.

In Theorem 1(3.1) (cf. also Figure 1) a certain scheme for the determination of complete scalar invariants of  $(X, G \times T)$  is organized,  $X = \text{seq } V_*$ . According to this, a scalar invariant  $f: X \rightarrow M$  is constructed in the form  $f = \varphi \circ h$ ,  $h: X \rightarrow X$  being a certain projection and  $\varphi: X \rightarrow M$  a complete scalar invariant of  $(X, G)$ . Furthermore, certain formulae are given by which  $\varphi$  and  $h$  can be obtained from  $\Phi: X^{(2)} \rightarrow M$ ,  $H: X^{(2)} \rightarrow X^{(2)}$  with analogical properties,  $X^{(2)}$  denoting a subset in  $X \times X$  including the graph of a certain mapping  $\iota: X \rightarrow X$ .

In Chapter II, certain mappings  $\Phi$  and  $\iota$  are constructed. Moreover, a certain global section for the space of all  $G$ -orbits in  $X$  is obtained (cf. Figures 2 and 3).

The next two chapters are concerned with the construction of a certain mapping  $H$ . The general idea is to obtain  $H$  as a morphism of some bundle  $(X^{(2)}, N^{(2)}, p_V(N \times N))$ ,  $p_V(N \times N)$  denoting a certain family of sets in  $N \times N$ , which assumes that the construction of  $H: X^{(2)} \rightarrow X^{(2)}$  will be reduced to that of  $H_A: X_A^{(2)} \rightarrow X_A^{(2)}$ ,  $X_A^{(2)}$  denoting the fibre over  $A$ .

For this purpose, the notions of direct sums of sets, matrices, elements in  $X$ ,  $X^{(2)}$  are studied in Chapter III where certain decompositions (maximal) are organized for them. In this way, the construction of  $H_A$  is reduced with  $A$  to the so-called connected sets.

In Chapter IV,  $H_A$  are reduced with  $A$  to be "normally-connected" and for such sets a certain algorithm for constructing  $H_A$  is proposed.

Let us note that all the constructions presented are effective and lead us to a certain strictly determined global surjective complete scalar invariant  $f$  of the type mentioned with  $M$  as a certain family of  $F$ -valued matrices ( $F$ -valued functions on  $N \times N$  with finite supports).

Four examples illustrating the method are contained in Chapter V.

The cardinality of a set  $A$  is denoted in the paper by  $\# A$ .

## I. Preliminaries

**§ 1. Remarks on  $g$ -spaces.** By a  $g$ -space we mean a pair  $(X, G)$  in which  $X$  is a set and  $G$  a group, acting on  $X$ .

For  $x \in X$  the set  $Gx = \{gx: g \in G\}$  is called the orbit through  $x$ .

With each  $g$ -space  $(X, G)$  is associated a morphism  $\tau: G \rightarrow \text{Aut } X$  of the group  $G$  into the group  $\text{Aut } X$  of all the automorphisms on  $X$ , defined by  $g \rightarrow \tau_g$ ,  $\tau_g(x) = gx$   $x \in X$   $g \in G$ .

The group  $\text{Ker } \tau$  let us call the group of effectivity of  $(X, G)$ . If the group of effectivity is trivial, the  $g$ -space is called effective.

By a  $K$ -space (Klein's space) we mean a pair  $(X, G)$  in which  $X$  is a set and  $G$  a group of transformations on  $X$ .

Of course, any  $K$ -space is a  $g$ -space and each effective  $g$ -space  $(X, G)$  is equivalent to a  $K$ -space  $(X, \text{im } \tau)$ , and therefore effective  $g$ -spaces we also call  $K$ -spaces.

A  $g$ -space  $(Y, H)$  is called a  $g$ -subspace of a given  $g$ -space  $(X, G)$  if  $Y \subset X$ ,  $H \subset G$  and the action of  $H$  on  $Y$  is the restriction of that of  $G$  on  $X$ .

Given are a  $g$ -space  $(X, G)$ , a set  $Y \subset X$  and a subgroup  $H \subset G$ .  $Y$  is called an  $H$ -invariant set in  $X$  if for all  $y \in Y$   $h \in H$  is  $hy \in Y$ . For each  $H$ -invariant set  $Y$  in  $X$ , with  $H$  acting on  $Y$  in the sense of the restriction of that of  $G$  on  $X$ ,  $(Y, H)$  is a  $g$ -subspace of  $(X, G)$ .

Given a  $g$ -space  $(X, G)$ , an arbitrary group  $H$  and an arbitrary morphism  $w: H \rightarrow G$ , we can define a  $g$ -space  $(X, H)$  by putting  $hx = w(h)x$   $x \in X$   $h \in H$ , which we also call a  $g$ -subspace of  $(X, G)$ .

With any  $g$ -space  $(X, G)$  may be associated the  $g$ -space  $(PX, G)$ ,  $PX$  denoting the family of all the subsets in  $X$ , with  $G$  acting on  $PX$  by  $gA = \{ga: a \in A\}$   $A \in PX$   $g \in G$ . The family  $pX$  of all the finite subsets in  $X$  is a  $G$ -invariant set in  $PX$  and therefore we get  $(pX, G)$  as a  $g$ -subspace of  $(PX, G)$ .

Take an arbitrary set  $N$  and a  $g$ -space  $(X, G)$ . Denote by  $X^N$  the family of all the mappings  $u: N \rightarrow X$  and by  $G^N$  the group of all the mappings  $q: N \rightarrow G$  with group operation being the multiplication of mappings i.e.  $qr$  is defined by  $(qr)(n) = q(n)r(n)$  ( $n \in N$ )  $q, r \in G^N$ . We get a  $g$ -space  $(X^N, G^N)$  defined by  $(qu)(n) = q(n)u(n)$  ( $n \in N$ )  $u \in X^N$   $q \in G^N$ . If we identify  $G$  with the subgroup in  $G^N$  consisting of all the constant mappings, then we get a  $g$ -subspace  $(X^N, G)$  of  $(X^N, G^N)$ .

We say that  $(X, G)$  commutes with  $(X, T)$ , if the action of  $G$  commutes with that of  $T$ .

If  $(X, G)$  commutes with  $(X, T)$ , then we can define a  $g$ -space  $(X, G \times T)$  by putting  $(g, t)x = g(tx) = t(gx)$   $x \in X$   $g \in G$   $t \in T$ .

Given  $g$ -spaces  $(X, G)$ ,  $(Y, H)$ , then we can define a  $g$ -space  $(X \times Y, G \times H)$  by  $(g, h)(x, y) = (gx, hy)$   $x \in X$   $y \in Y$   $g \in G$   $h \in H$ , called the product of the given spaces.

In the case when  $G = H$ , we may define  $(X \times Y, G)$  as the  $g$ -subspace of  $(X \times Y, G \times G)$  induced by the morphism  $\Delta: G \rightarrow G \times G$ ;  $g \rightarrow (g, g)$ .

In the case when  $X = Y$ , the  $g$ -space  $(X \times X, G)$  is identical with the  $g$ -space  $(X^N, G)$  for  $N$  such that  $\#N = 2$ ,  $\#N$  denoting the cardinality of  $N$ .

By an invariant (morphism, concomitant) of a given pair  $((X, G), (Y, H))$  of  $g$ -spaces is meant a pair  $(f, \lambda)$  of a mapping  $f: X \rightarrow Y$  and a group morphism  $\lambda: G \rightarrow H$  satisfying the condition  $f(gx) = \lambda(g)f(x)$  for all  $x \in X$   $g \in G$ . In the case when  $G = H$  and  $\lambda = id_G$ , we call  $f$  itself an invariant and we then speak of a  $G$ -invariant.

If each orbit of a  $g$ -space  $(Y, H)$  is a one-element set in  $Y$ , we call  $(Y, H)$  a scalar  $g$ -space.

In the case when  $(Y, H)$  is a scalar  $g$ -space, any invariant  $(f, \lambda)$  of the pair  $((X, G), (Y, H))$  is called a scalar invariant. In this case, if  $\ker \lambda = G$ , we call  $f$  itself a scalar invariant.

Let  $(f, \lambda)$  be an invariant of  $((X, G), (Y, H))$ . Then  $(\text{im} f, \text{im} \lambda)$  is a  $g$ -subspace of  $(Y, H)$ .

Given a mapping  $h: X \rightarrow X$  and an equivalence relation  $R$  on  $X$ , we say  $R$  is consistent with  $h$  if each class of equivalency of  $R$  is transformed by  $h$  into a class of  $R$ , i.e. if the following is satisfied for all  $x, y \in X$

$$xRy \Rightarrow h(x)R h(y).$$

Given mappings  $\varphi: X \rightarrow M$  and  $h: X \rightarrow X$ , we say  $\varphi$  is consistent with  $h$  if the relation  $R$ , defined on  $X$  by  $xRy$  iff  $\varphi(x) = \varphi(y)$ , is consistent with  $h$ , or, if for all  $x, y \in X$

$$\varphi(x) = \varphi(y) \Rightarrow \varphi(h(x)) = \varphi(h(y)).$$

In other words,  $\varphi$  is consistent with  $h$  if each  $\varphi$ -fibre is transformed by  $h$  into a  $\varphi$ -fibre. By a  $\varphi$ -fibre we mean any set  $\varphi^{-1}(a)$  for  $a \in \text{im} \varphi$ .

Let  $\varphi: X \rightarrow M$  be a surjection consistent with a given mapping  $h: X \rightarrow X$ . Then there is a mapping  $\chi: M \rightarrow M$  uniquely determined by the condition

$$(1.1.1) \quad \chi \circ \varphi = \varphi \circ h$$

and we say that  $\chi$  is induced from  $h$  by  $\varphi$ .

Given a  $g$ -space  $(X, T)$  and a mapping  $\varphi: X \rightarrow M$ , we say  $\varphi$  is consistent with the action of  $T$ , or shortly with  $T$ , if for each  $t \in T$   $\varphi$  is consistent with the  $h_t: X \rightarrow X$  defined by  $h_t(x) = tx$   $x \in X$ , or, if for all  $x, y \in X$   $t \in T$

$$\varphi(x) = \varphi(y) \Rightarrow \varphi(tx) = \varphi(ty).$$

Any injection  $\varphi$  is consistent with  $T$ .

Given a  $g$ -space  $(X, T)$  and a surjection  $\varphi: X \rightarrow M$  consistent with  $T$ , then we get a  $g$ -space  $(M, T)$  defined by  $ta = \varphi(tx)$ ,  $a = \varphi(x)$   $t \in T$ . The independence of the choice of  $x$  is evident. We say that  $(M, T)$  is induced from  $(X, T)$  by  $\varphi$ .

If  $\varphi$  is a bijection, then  $(M, T)$  exists.

The given  $g$ -spaces  $(X, G)$ ,  $(Y, H)$  are called equivalent if there exists an invariant  $(f, \lambda)$  with bijections  $f$  and  $\lambda$ .

Take a  $g$ -space  $(X, G)$  and a normal subgroup  $H \subset G$ . Assuming  $H$  is contained in the group of effectivity of  $(X, G)$  we may define a  $g$ -space  $(X, G/H)$  by  $[g]x = gx$   $x \in X$   $g \in G$ . In the case when  $H$  itself is the group of effectivity of  $(X, G)$ ,  $(X, G/H)$  is effective, i.e. is a  $K$ -space.

Given a  $g$ -space  $(X, G)$  and an equivalence relation  $R$  on  $X$  consistent with  $G$ , then we get a  $g$ -space  $(X/R, G)$  defined by  $g[x] = [gx]$   $x \in X$   $g \in G$ .

Given a  $g$ -space  $(X, G)$  and an arbitrary normal subgroup  $H \subset G$ , then  $H$  induces an equivalence relation  $H$  on  $X$  defined by  $xHx'$  iff  $x' = hx$  for some  $h \in H$ . This relation is consistent with  $G$  and we can form a  $g$ -space  $(X/H, G)$ . In the case when  $(X, G)$  is a  $K$ -space, the group of effectivity of  $(X/H, G)$  is  $H$  and we can form a  $g$ -space  $(X/H, G/H)$  which is a  $K$ -space.

Take a  $g$ -space  $(X, G)$ , a normal subgroup  $H \subset G$  and an equivalence relation  $R$  on  $X$ . Assuming that  $R$  is consistent with  $G$  and that  $H$  is contained in the group of effectivity of  $(X/R, G)$  we may form the  $g$ -space  $(X/R, G/H)$  called the factor  $g$ -space of  $(X, G)$  with respect to  $R$  and  $H$ .

Given a mapping  $f: X \rightarrow Y$  and equivalence relations  $R$  and  $S$  on  $X$  and  $Y$ , respectively, we say  $R$  and  $S$  are consistent with  $f$ , if each class of  $R$  is transformed by  $f$  into a class of  $S$ , i.e. if  $xRx' \Rightarrow f(x)Sf(x')$  for all  $x, x' \in X$ .

Let  $R, S$  be consistent with  $f$ . Then we can define a mapping  $\tilde{f}: X/R \rightarrow Y/S$ ;  $[x]_R \rightarrow [f(x)]_S$  called the factorization of  $f$  with respect to  $R, S$ .

Let  $(f, \lambda)$  be an invariant of  $g$ -spaces  $(X, G), (Y, H)$ . Then the relations  $R$  and  $S$  defined on  $X$  and on  $Y$  by:  $xRx' \text{ iff } x' \in Gx$  and  $ySy' \text{ iff } y' \in Hy, H = \text{im } \lambda$ , are consistent with  $f$ . Denoting by  $\text{orb } X$  and  $\lambda\text{-orb } Y$  the spaces of all the  $G$ -orbits in  $X$  and  $H$ -orbits in  $Y$ , respectively, we get the factorization  $\lambda\text{-}\tilde{f}$ :  $\text{orb } X \rightarrow \lambda\text{-orb } Y; Gx \rightarrow Hf(x)$ , called the  $\lambda$ -orbital factorization of  $(f, \lambda)$ .

We can also define another factorization of  $(f, \lambda)$ ,  $\tilde{f}: \text{orb } X \rightarrow \text{orb } Y; Gx \rightarrow Hf(x)$ , called orbital. In the case when  $\lambda$  is surjective,  $\lambda\text{-}\tilde{f}$  and  $\tilde{f}$  coincide. They also coincide for scalar invariants.

An invariant  $(f, \lambda)$  is called  $\lambda$ -complete if its factorization  $\lambda\text{-}\tilde{f}$  is injective, and  $(f, \lambda)$  is called complete if  $\tilde{f}$  is injective. As a special case we get the notion of a complete scalar invariant. For each surjective complete scalar invariant  $(f, \lambda)$  the orbital factorization  $\tilde{f}$  is a bijection. This means that  $f^{-1}(y)$  is a  $G$ -orbit for each  $y \in Y$  and therefore  $\text{orb } X$  is characterized by  $Y$ , or by  $(Y, H)$ .

A given  $g$ -space  $(X, G)$  is called  $\lambda$ -orbitally (orbitally) equivalent to a  $g$ -space  $(Y, H)$ , if there exists a  $\lambda$ -complete (complete) invariant  $(f, \lambda)$ ,  $f: X \rightarrow Y \lambda: G \rightarrow H$ , with  $f$  surjective; then  $\lambda\text{-}\tilde{f}$  ( $\tilde{f}$ ) is a bijection.

The notions of complete invariants and orbital equivalency for  $g$ -spaces are of great significance in the problem of classification of  $g$ -spaces (or  $K$ -spaces). The classification of  $g$ -spaces is understood in the following sense.

By a classification of a  $g$ -space  $(X, G)$ , or  $G$ -classification of  $X$ , we mean the decomposition of  $X$  into its disjoint  $G$ -invariant subsets. If all the components are  $G$ -orbits, the classification is called complete.

We are interested in obtaining complete classifications for some  $K$ -spaces which will be introduced in the next paragraph. But what does it mean to say that the complete classification for a given  $g$ -space  $(X, G)$  is obtained? Three points of view may be distinguished.

1. If we need to know whether the two given elements  $x, x' \in X$  lie in a  $G$ -orbit (we say, are  $G$ -congruent or congruent) or not, then any complete scalar  $G$ -invariant is a good criterion for this.

2. If we need to point out  $G$ -orbits by their representative elements, then we have to give a section of the space of the orbits.

3. If we need a method for reconstruction of both sections of orbits and

