

A statistical approach to the heat equation

by M. CAPIŃSKI

In [1] C. Foias has introduced the notion of a statistical solution of the Navier-Stokes equations. In this paper we apply his methods to the heat equation, giving detailed proofs of the existence (Th. 4) and global uniqueness (Th. 7) theorems.

1. Introduction

1.1. Let Ω be a bounded domain in R^n with boundary of class C^2 . The Hilbert space of measurable real valued functions u defined on Ω with a finite norm

$$\|u\| := \left(\int_{\Omega} |u(x)|^2 dx \right)^{1/2}$$

will be denoted by L^2 . We put

$$\int_{\Omega} u(x)v(x)dx =: (u, v) \quad \text{for } u, v \in L^2.$$

We define the subspace H_0^1 of L^2 in the following way: H_0^1 is the closure of $C_0^\infty(\Omega)$ in H^1

where $H^1 = \left\{ u \in L^2 : \frac{\partial u}{\partial x_k} \in L^2 (k = 1, \dots, n) \right\}$ (derivatives are taken in the sense of the

theory of distributions) with norm $\|u\|_{H^1} = \left(\|u\|^2 + \sum_{k=1}^n \left\| \frac{\partial u}{\partial x_k} \right\|^2 \right)^{1/2}$. H_0^1 is a Hilbert

space with a scalar product

$$(u, v)_1 := \sum_{k=1}^n \left(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k} \right) \quad \text{for } u, v \in H_0^1.$$

We denote $\|u\|_1 := ((u, u)_1)^{1/2}$ for $u \in H_0^1$.

1.2. Let us consider the operator $-\Delta$ defined on $C_0^\infty(\Omega)$. There is a unique self-adjoint extension A of $-\Delta$ such that

$$\begin{aligned} D(A) &\subset H_0^1, \\ D(A^{1/2}) &= H_0^1, \\ (u, v)_1 &= (A^{1/2}u, A^{1/2}v), \\ \|u\| &\leq c_1 \|u\|_1 \end{aligned}$$

for $u, v \in H_0^1$ (see [2] p. 191, [3] p. 110, [4] p. 24).

1.3. For any fixed $u \in L^2$ applying the Riesz theorem for a functional $(u, \cdot): H_0^1 \rightarrow \mathbb{R}$ we get a unique $Iu \in H_0^1$ satisfying

$$(u, v) = (Iu, v)_1 \quad \text{for } v \in H_0^1.$$

It is easy to see that $I: L^2 \rightarrow H_0^1$ is linear, continuous and such that $\|Iu\|_1 \leq c_1 \|u\|$ for $u \in L^2$. The complete extension of the unitary space $(L^2, (\cdot, \cdot)_{-1})$ where

$$(u, v)_{-1} := (Iu, v) = (Iu, Iv)_1$$

will be denoted by H_0^{-1} . We have

$$\begin{aligned} H_0^1 &\subset L^2 \subset H_0^{-1}, \\ \|u\|_{-1} &\leq c_1 \|u\| \quad \text{for } u \in L^2. \end{aligned}$$

Extending continuously I on H_0^{-1} we get the bijection $I: H_0^{-1} \rightarrow H_0^1$ such that

$$(\alpha, \beta)_{-1} = (I\alpha, I\beta)_1 \quad \text{for } \alpha, \beta \in H_0^{-1}.$$

For $\alpha \in H_0^{-1}$, $u \in H_0^1$ we put

$$\alpha[u] := \lim_{n \rightarrow \infty} (v_n, u)$$

where $v_n \rightarrow \alpha$, $v_n \in H_0^1$. We have the following properties

$$\begin{aligned} |\alpha[u]| &\leq \|\alpha\|_{-1} \|u\|_1, \\ \alpha[u] &= (I\alpha, u)_1. \end{aligned}$$

H_0^{-1} can be identified in the following sense with a set of all continuous linear functionals on H_0^1 . For $\alpha \in H_0^{-1}$ $l_\alpha(u) := \alpha[u]$ is a linear continuous functional $l_\alpha: H_0^1 \rightarrow \mathbb{R}$, $\|\alpha\|_{-1} = \|l_\alpha\| := \sup_{u \in H_0^1} \frac{|l_\alpha(u)|}{\|u\|_1}$. Conversely for $l: H_0^1 \rightarrow \mathbb{R}$ linear and continuous there exists a unique $v_l \in H_0^1$ such that $l(u) = (v_l, u)_1 = I^{-1}v_l[u]$. We put $\alpha_l := I^{-1}v_l \in H_0^{-1}$ obtaining $\|l\| = \|v_l\|_1 = \|\alpha_l\|_{-1}$. (for details see [5] ch. I).

1.4. We extend the operator A on the whole H_0^1 . For $u \in H_0^1$ we put

$$A_e u := I^{-1}u \in H_0^{-1}.$$

We have

$$A_e u[v] = (u, v)_1 \quad \text{for any } v \in H_0^1.$$

For $u \in D(A)$, $v \in H_0^1$ we get

$$(A_\epsilon u, v)_{-1} = (IA_\epsilon u, Iv)_1 = (u, Iv)_1 = (A^{1/2}u, A^{1/2}Iv) = (Au, Iv) = (Au, v)_{-1}$$

hence $A_\epsilon u = Au$.

1.5. In [6] (p. 163) the following theorem may be found:

THEOREM 1. Let us fix any $f \in L^2(0, T; L^2)$ where $T > 0$ or $T = +\infty$. For each $u_0 \in L^2$ there exists exactly one $u: [0, T) \rightarrow L^2$ such that

- 1) $u(t) \in H_0^1$ for $t \in (0, T)$,
- 2) there exists a constant c_2 such that

$$(1) \quad \text{ess sup}_{t \in [0, T)} \|u(t)\| + \left(\int_0^T \|u(t)\|_1^2 dt \right)^{1/2} \leq c_2 \left(\frac{1}{2} \|u_0\| + \int_0^T \|f(t)\| dt \right),$$

$$(2) \quad 3) \quad (u(\tau), v(\tau)) - (u_0, v(0)) - \int_0^\tau (u(t), v'_t(t)) dt + \int_0^\tau (u(t), v(t))_1 dt = \int_0^\tau (f(t), v(t)) dt$$

holds for all $\tau \in [0, T)$, $v \in C(0, T; H_0^1)$ such that there exists $v'_t \in L^2(0, T; L^2)$,

4) the mapping $S: [0, T) \times L^2 \rightarrow L^2$ defined as follows $S(t, u_0) := u(t)$ is continuous with respect to each variable.

1.6. We shall prove the following corollary of Th. 1.

THEOREM 2. Let $f \in L^2(0, T; L^2)$. For every $u_0 \in L^2$ there exists exactly one $u: [0, T) \rightarrow L^2$ such that

- 1) $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1)$,
- 2) u has a weak derivative u' (for definition see [7] p. 387) as a function $u: [0, T) \rightarrow H_0^{-1}$,
- 3) $\begin{cases} u'(t) + A_\epsilon u(t) = f(t) & \text{for almost all } t \in (0, T) \\ u(0) = u_0, \end{cases}$
- 4) there exist constants c_3, \dots, c_6 such that

$$(3) \quad \begin{cases} \|u(t)\|^2 \leq c_3 \|u_0\|^2 + c_4 & \text{a.e. on } (0, T) \\ \int_0^T \|u(t)\|_1^2 dt \leq c_5 \|u_0\|^2 + c_6. \end{cases}$$

Proof: We take u obtained in Th. 1. For every $v \in C_0(0, T; H_0^{-1})$ having integrable $v'_t: (0, T) \rightarrow H_0^{-1}$ we have

$$-\int_0^T (u(t), v'_t(t))_{-1} dt = -\int_0^T (u(t), I(v'_t(t))) dt.$$

It is easy to see that $(I \circ v)'$ exists and that $(I \circ v)' = I \circ v'$. This yields

$$\begin{aligned} -\int_0^T (u(t), I(v'_t(t))) dt &= -\int_0^T (u(t), (I \circ v)'(t)) dt \\ &= \int_0^T [-(u(t), I \circ v(t))_1 + (f(t), I \circ v(t))] dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^T [-(A_e u(t), v(t))_{-1} + (f(t), v(t))_{-1}] dt \\
&= \int_0^T (-A_e u(t) + f(t), v(t))_{-1} dt
\end{aligned}$$

thus u has a weak derivative u' and $u'(t) = -A_e u(t) - f(t)$ a.e. on $(0, T)$. Properties 1), 4) are simple consequences of (1); thus theorem 2 is proved.

2. Statistical solutions of the heat equation

2.1. Let us fix μ_0 — any Borel, probability measure on L^2 such that

$$(4) \quad \int_{L^2} \|u\|^2 d\mu_0(u) < +\infty.$$

For any fixed $t \in (0, T)$ $S(t, \cdot): L^2 \rightarrow H_0^1$. We put

$$\mu_t(\omega) := \mu_0(S(t, \cdot)^{-1}(\omega))$$

for any Borel subset of L^2 .

Definition 1. A family $\{\mu_t\}_{t \in (0, T)}$ of Borel, probability measures on L^2 will be called *basic* if

$$1) \quad \int_{H_0^1} \|u\|^2 d\mu_t(u) \text{ is an essentially bounded function of } t \in (0, T)$$

$$\text{(shortly } \int_{H_0^1} \|u\|^2 d\mu_t(t) \in L^\infty(0, T)),$$

$$2) \quad \int_{H_0^1} \|u\|_1^2 d\mu_t(u) \in L^1(0, T),$$

$$3) \quad \mu_t(L^2 \setminus H_0^1) = 0 \quad \text{for } t \in (0, T).$$

THEOREM 3. *The family of measures defined above is basic.*

Proof: We rewrite the inequalities (3) in the following form

$$\|S(t, u_0)\|^2 \leq c_3 \|u_0\|^2 + c_4 \quad \text{a.e. on } (0, T)$$

$$\int_0^T \|S(t, u_0)\|_1^2 dt \leq c_5 \|u_0\|^2 + c_6.$$

Hence

$$\int_{L^2} \|S(t, u)\|^2 d\mu_0(u) \leq c_3 \int_{L^2} \|u\|^2 d\mu_0(u) + c_4 \leq c_7 \quad \text{a.e. on } (0, T),$$

$$\int_{L^2} \int_0^T \|S(t, u)\|_1^2 dt d\mu_0(u) \leq c_5 \int_{L^2} \|u\|^2 d\mu_0(u) + c_6 \leq c_8,$$

which yields

$$\int_0^T \int_{L^2} \|S(t, u)\|_1^2 d\mu_0(u) dt = \int_{L^2} \int_0^T \|S(t, u)\|_1^2 dt d\mu_0(u) \leq c_8.$$

For any mapping $\psi: H_0^1 \rightarrow R$ if $\psi \circ S(t, \cdot)$ is μ_0 integrable then ψ is μ_t integrable and

$$(5) \quad \int_{H_0^1} \psi(u) d\mu_t(u) = \int_{L^2} \psi(S(t, u)) d\mu_0(u) \quad \text{for } t \in (0, T).$$

We can apply this well-known fact, obtaining

$$\int_{H_0^1} \|u\|^2 d\mu_t(u) \leq c_7 \quad \text{a.e. on } (0, T),$$

$$\int_0^T \int_{H_0^1} \|u\|_1^2 d\mu_t(u) dt \leq c_8,$$

which finishes the proof of the first and second parts of Def. 1. At the end we notice that $\mu_t(L^2 \setminus H_0^1) = \mu_0(S(t, \cdot)^{-1}(L^2 \setminus H_0^1)) = \mu_0(\emptyset) = 0$ for all $t \in (0, T)$, thus theorem 3 is proved.

2.2. We define the set \mathcal{F}^{ind} of *time independent test functionals* in the following way:

Definition 2. We say that continuous functional $\varphi: L^2 \rightarrow R$ belongs to \mathcal{F}^{ind} if

1) there exist constants c_9, c_{10} such that

$$(6) \quad |\varphi(u)| \leq c_9 + c_{10} \|u\| \quad \text{for } u \in L^2,$$

2) for every $u \in L^2$ there exists $\varphi'(u) \in H_0^1$ such that

$$\frac{|\varphi(u+v) - \varphi(u) - (\varphi'(u), v)|}{\|v\|} \rightarrow 0 \quad \text{as } \|v\| \rightarrow 0, v \in H_0^1;$$

3) $\varphi': L^2 \rightarrow H_0^1$ is a continuous mapping,

4) $\|\varphi'(u)\|_1 \leq c_{11}$ for all $u \in L^2$ (constants c_9, c_{10}, c_{11} depend on φ).

$\{w_i\}_{i=1,2,\dots}$ will denote the orthonormal basis of L^2 such that $Aw_i = \lambda_i w_i$ for some $\lambda_i \geq 0$ ($i = 1, 2, \dots$). P_m denotes the orthogonal projection of L^2 on the subspace spanned by $\{w_1, \dots, w_m\}$ (for $m = 0$ $P_m = 0$).

Definition 3. $\varphi \in \mathcal{F}^{\text{ind}}$ will be called *elementary* ($\varphi \in \mathcal{F}_0^{\text{ind}}$) if for some m (depending on φ) we have $\varphi(u) = \varphi(P_m u)$ for all $u \in L^2$.

2.3. We introduce the notion of a statistical solution of the heat equation only in terms of this equation: the solutions of heat equation are not used here.

Definition 4. A family of measures $\{\mu_t\}_{t \in (0, T)}$ will be called *a statistical solution of the heat equation* with an initial measure μ_0 if

1) $\{\mu_t\}_{t \in (0, T)}$ is basic,

2) for all $\varphi \in \mathcal{F}^{\text{ind}}$, $\tau \in B \subset (0, T)$ we have

$$(7) \quad \int_{H_0^1} \varphi(u) d\mu_\tau(u) - \int_{L^2} \varphi(u) d\mu_0(u) - \int_0^\tau \int_{H_0^1} (u, \varphi'(u))_1 d\mu_t(u) dt$$

$$= \int_0^\tau \int_{H_0^1} (f(t), \varphi'(u)) d\mu_t(u) dt,$$

where B does not depend on φ and is a set of full measure in $(0, T)$.

Remark. All results of this paper remain valid if we take another definition of the statistical solution, namely

Definition 4'. A family of measures $\{\mu_t\}_{t \in (0, T)}$ will be called a *statistical solution* of the heat equation with an initial measure μ_0 if

- 1) $\{\mu_t\}_{t \in (0, T)}$ is basic,
- 2) (7) holds for all $\tau \in (0, T)$.

LEMMA 1. If (7) holds for every $\varphi \in \mathcal{F}_0^{\text{ind}}$ then it holds also for all $\varphi \in \mathcal{F}^{\text{ind}}$.

Proof: We take any $\varphi \in \mathcal{F}^{\text{ind}}$ and we put $\varphi_m(u) := \varphi(P_m u)$ ($m = 1, 2, \dots$). We shall show that

$$(8) \quad \varphi_m \in \mathcal{F}_0^{\text{ind}} \quad \text{for all } m.$$

$$\varphi_m(P_m u) = \varphi(P_m^2 u) = \varphi(P_m u) = \varphi_m(u).$$

We put $\varphi'_m(u) := P_m \varphi'(P_m u)$. Of course $\varphi'_m(u) \in H_0^1$.

$$\frac{|\varphi_m(u+v) - \varphi_m(u) - (P_m \varphi'(P_m u), v)|}{\|v\|} \leq \frac{|\varphi(P_m u + P_m v) - \varphi(P_m u) - (\varphi'(P_m u), P_m v)|}{\|P_m v\|} \rightarrow 0$$

when $\|v\| \rightarrow 0$, $v \in H_0^1$.

$$\|\varphi'_m(u)\|_1 = \|P_m \varphi'(P_m u)\|_1 \leq \|\varphi'(P_m u)\|_1 \leq c_{11}$$

and φ'_m is obviously continuous, thus (8) is proved.

We denote $\psi_\varphi(t, u) := -(u, \varphi'(u))_1 + (f(t), \varphi'(u))$. We shall show that

$$(9) \quad \psi_{\varphi_m}(t, u) \rightarrow \psi_\varphi(t, u) \quad \text{as } m \rightarrow \infty$$

for all $(t, u) \in (0, T) \times H_0^1$.

$$|\psi_{\varphi_m}(t, u) - \psi_\varphi(t, u)| \leq |(u, \varphi'_m(u) - \varphi'(u))_1| + |(f(t), \varphi'_m(u) - \varphi'(u))|$$

$$\leq \|u\|_1 \cdot \|\varphi'_m(u) - \varphi'(u)\|_1 + \|f(t)\| \cdot c_1 \cdot \|\varphi'_m(u) - \varphi'(u)\|_1;$$

$$\|\varphi'_m(u) - \varphi'(u)\|_1 = \|P_m \varphi'(P_m u) - \varphi'(u) \pm P_m \varphi'(u)\|_1 \leq \|P_m \varphi'(P_m u) - P_m \varphi'(u)\|_1 +$$

$$+ \|P_m \varphi'(u) - \varphi'(u)\|_1 \leq \|\varphi'(P_m u) - \varphi'(u)\|_1 + \|P_m \varphi'(u) - \varphi'(u)\|_1 \rightarrow 0$$

as $m \rightarrow \infty$ because it may easily be verified that $\|P_m v - v\|_1 \rightarrow 0$ as $m \rightarrow \infty$ for any $v \in H_0^1$.

Thus (9) is proved. Now we shall demonstrate, that $\int_0^\tau \int_{H_0^1} \psi_\varphi(t, u) d\mu_t(u) dt$ makes sense and that

$$(10) \quad \int_0^\tau \int_{H_0^1} \psi_{\varphi_m}(t, u) d\mu_t(u) dt \rightarrow \int_0^\tau \int_{H_0^1} \psi_\varphi(t, u) d\mu_t(u) dt$$

as $m \rightarrow \infty$, for each $\tau \in (0, T)$.

$$|\psi_{\varphi_m}(t, u)| \leq \|u\|_1 \cdot c_{11} + \|f(t)\| \cdot c_1 \cdot c_{11}.$$

The sequence $\psi_{\varphi m}$ has a μ_t integrable majorant for almost all $t \in (0, T)$ so that in virtue of Lebesgue's theorem on dominated convergence $\int_{H_0^1} \psi_{\varphi}(t, u) d\mu_t(u)$ makes sense and

$$\int_{H_0^1} \psi_{\varphi m}(t, u) d\mu_t(u) \rightarrow \int_{H_0^1} \psi_{\varphi}(t, u) d\mu_t(u)$$

as $m \rightarrow \infty$, for almost all $t \in (0, T)$.

$$|\int_{H_0^1} \psi_{\varphi m}(t, u) d\mu_t(u)| \leq c_{12} \int_{H_0^1} \|u\|_1^2 d\mu_t(u) + c_{13} \|f(t)\|^2.$$

The majorant is t integrable on $(0, T)$ and we may apply Lebesgue's theorem once more. Thus (10) is proved. Similarly we can prove that $\int_{H_0^1} \varphi(u) d\mu_t(u)$, $\int_{L^2} \varphi(u) d\mu_0$ make sense and

that

$$\begin{aligned} \int_{H_0^1} \varphi_m(u) d\mu_t(u) &\rightarrow \int_{H_0^1} \varphi(u) d\mu_t(u) \quad \text{as } m \rightarrow \infty, \\ \int_{L^2} \varphi_m(u) d\mu_0(u) &\rightarrow \int_{L^2} \varphi(u) d\mu_0(u) \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Thus Lemma 1. is proved.

THEOREM 4. *The family of measures defined in sec. 2.1 is a statistical solution of the heat equation with initial measure μ_0 .*

Proof. We take any $\varphi \in \mathcal{F}_0^{\text{ind}}$. It is easy to show that $\varphi'(u) = P_m \varphi'(P_m u)$ where m is such that $\varphi(u) = \varphi(P_m u)$. We extend P_m to $P_{m_e}: H_0^{-1} \rightarrow H_0^1$ putting

$$P_{m_e} \alpha = \alpha [w_1] w_1 + \dots + \alpha [w_m] w_m \quad \text{for } \alpha \in H_0^{-1}.$$

We have

$$\varphi(u(t)) = \varphi(P_{m_e}(u(t))) \quad \text{for } t \in (0, T).$$

We shall show that $\varphi \circ P_{m_e} \circ u$ has a weak derivative and that

$$(11) \quad \frac{d}{dt} (\varphi \circ P_{m_e} \circ u)(t) = (\varphi'(P_{m_e} u(t)), P_{m_e} u'(t))$$

We take any non-negative, symmetric $\varrho \in C^\infty(R)$ such that $\varrho(t) = 0$ if $|t| \geq 1$ and $\int_{-1}^1 \varrho(t) dt = 1$.

For $g \in L^2(0, T; H_0^{-1})$ we put $g_\varepsilon(t) = \int_0^T \varrho_\varepsilon(|t-\tau|) g(\tau) d\tau$ where $\varrho_\varepsilon(t) = \frac{1}{\varepsilon} \left(\frac{t}{\varepsilon}\right)$. $u: (0, T) \rightarrow H_0^{-1}$ has a weak derivative u' thus (cf [8] p. 116) for each $0 < a < b < T$

$$u'_\varepsilon = (u')_\varepsilon,$$

$$u_\varepsilon \rightarrow u,$$

$$u'_\varepsilon \rightarrow u',$$

as $\varepsilon \rightarrow 0$: convergence in $L^2(a, b; H_0^{-1})$. $\varphi \circ P_{m_e}$ is a continuous mapping from H_0^{-1} to R thus

$$\varphi(P_{m_e} u_\varepsilon(t)) \rightarrow \varphi(P_{m_e} u(t))$$

as $\varepsilon \rightarrow 0$, for almost $t \in (0, T)$. $\varphi \circ P_{m_\varepsilon} \circ u_\varepsilon \in C^1(0, T; R)$ and

$$\frac{d}{dt}(\varphi \circ P_{m_\varepsilon} \circ u_\varepsilon)(t) = (\varphi'(P_{m_\varepsilon} u_\varepsilon(t)), P_{m_\varepsilon} u'_\varepsilon(t)).$$

We have

$$(\varphi'(P_{m_\varepsilon} u_\varepsilon(t)), P_{m_\varepsilon} u'_\varepsilon(t)) \rightarrow (\varphi'(P_{m_\varepsilon} u(t)), P_{m_\varepsilon} u'(t))$$

as $\varepsilon \rightarrow 0$ a.e. on $(0, T)$ and moreover

$$|(\varphi'(P_{m_\varepsilon} u_\varepsilon(t)), P_{m_\varepsilon} u'_\varepsilon(t))| \leq c_{11} \left| \sum_{i=1}^m u'_\varepsilon(t) [w_i] \cdot w_i \right| \leq c_{11} \|u'_\varepsilon(t)\|_{-1}.$$

Now we take any $h \in C_0^1(0, T)$ and we have

$$\int_0^T \frac{d}{dt}(\varphi \circ P_{m_\varepsilon} \circ u_\varepsilon)(t) h(t) dt = - \int_0^T (\varphi \circ P_{m_\varepsilon} \circ u_\varepsilon)(t) h'(t) dt.$$

Letting $\varepsilon \rightarrow 0$ we get

$$\int_0^T (\varphi'(P_{m_\varepsilon} u(t)), P_{m_\varepsilon} u'(t)) h(t) dt = - \int_0^T (\varphi \circ P_{m_\varepsilon} \circ u)(t) h'(t) dt$$

thus (11) is proved.

$$\begin{aligned} (\varphi'(P_m u(t)), P_m u'(t)) &= (\varphi'(P_m u(t)), \sum_{i=1}^m u'(t) [w_i] \cdot w_i) \\ &= \sum_{i=1}^m (\varphi'(P_m u(t)), (Iu'(t), w_i)_1 w_i) \\ &= \sum_{i=1}^m (Iu'(t), w_i)_1 (\varphi'(P_m u(t)), w_i) \\ &= \sum_{i=1}^m (Iu'(t), (\varphi'(P_m u(t)), w_i) w_i)_1 \\ &= (Iu'(t), \sum_{i=1}^m (\varphi'(P_m u(t)), w_i) w_i)_1 \\ &= (Iu'(t), P_m \varphi'(P_m u(t)))_1 \\ &= (Iu'(t), \varphi'(u(t)))_1 \\ &= u'(t) [\varphi'(u(t))]. \end{aligned}$$

Finally we get for almost all $t \in (0, T)$

$$(12) \quad \frac{d}{dt} \varphi(u(t)) = u'(t) [\varphi'(u(t))].$$

$$\left| \frac{d}{dt} \varphi(u(t)) \right| \leq \|u'(t)\|_{-1} \cdot \|\varphi'(u(t))\|_1 \leq \|A_\varepsilon u(t)\|_{-1} c_{11} + \|f(t)\|_{-1} c_{11}$$

$$\leq \|u(t)\|_1 c_{11} + \|f(t)\|_1 c_{11} \quad \text{a.e. on } (0, T).$$

The right side of this inequality is t integrable on $(0, T)$ and

$$(13) \quad \int_0^T \left| \frac{d}{dt} \varphi(u(t)) \right| dt \leq c_{11} \int_0^T \|u(t)\|_1 dt + c_1 c_{11} \int_0^T \|f(t)\| dt \leq c_{14} + c_{15} \|u_0\|^2.$$

We rewrite this last inequality in the following form

$$\int_0^T \left| \frac{d}{dt} \varphi(S(t, u)) \right| dt \leq c_{14} + c_{15} \|u\|^2 \quad \text{for all } u \in L^2.$$

The right side is μ_0 integrable, hence

$$\int_{L^2} \int_0^T \left| \frac{d}{dt} \varphi(S(t, u)) \right| dt d\mu_0(u) < +\infty.$$

From Fubini's theorem

$$g(t) := \int_{L^2} \frac{d}{dt} \varphi(S(t, u)) d\mu_0(u)$$

is t integrable on $(0, T)$. We shall prove that

$$h(t) := \int_{L^2} \varphi(S(t, u)) d\mu_0(u)$$

has a weak derivative g for almost all $t \in (0, T)$. We take any $r: (0, T) \rightarrow \mathbb{R}$ of class C^1 with compact support. We get

$$\begin{aligned} - \int_0^T r'(t) h(t) dt &= - \int_0^T \int_{L^2} r'(t) \varphi(S(t, u)) d\mu_0(u) dt \\ &= \int_{L^2} \left[- \int_0^T r'(t) \varphi(S(t, u)) dt \right] d\mu_0(u) \\ &= \int_{L^2} \int_0^T r(t) \frac{d}{dt} \varphi(S(t, u)) dt d\mu_0(u) \\ &= \int_0^T r(t) g(t) dt \end{aligned}$$

applying Fubini's theorem twice by means of (4), (6), (13). Finally we have

$$\frac{d}{dt} \int_{L^2} \varphi(S(t, u)) d\mu_0(u) = \int_{L^2} \frac{d}{dt} \varphi(S(t, u)) d\mu_0(u) \quad \text{a.e. on } (0, T).$$

From (12) we get

$$\begin{aligned} \frac{d}{dt} \varphi(S(t, u)) &= \frac{\partial}{\partial t} S(t, u) [\varphi'(S(t, u))] = (-A_e S(t, u) + f(t)) [\varphi'(S(t, u))] \\ &= -(S(t, u), \varphi'(S(t, u)))_1 + (f(t), \varphi'(S(t, u))) \end{aligned}$$

so that

$$\begin{aligned} \frac{d}{dt} \int_{L^2} \varphi(S(t, u)) d\mu_0(u) + \int_{L^2} (S(t, u), \varphi'(S(t, u)))_1 d\mu_0(u) \\ = \int_{L^2} (f(t), \varphi'(S(t, u))) d\mu_0(u) \quad \text{a.e. on } (0, T). \end{aligned}$$

Applying (5) we get

$$\frac{d}{dt} \int_{H_0^1} \Phi(u) d\mu_t(u) + \int_{H_0^1} (u, \varphi'(u))_1 d\mu_t(u) = \int_{H_0^1} (f(t), \varphi'(u)) d\mu_t(u) \quad \text{a.e. on } (0, T).$$

Integrating this last equality from 0 to τ with respect to t for almost all $\tau \in (0, T)$ we get the result; thus theorem 4 is proved.

Remark. We may simply prove that (7) holds for all $\tau \in (0, T)$. It suffices to observe that $\int_{H_0^1} \varphi(u) d\mu_t(u) = \int_{L^2} \varphi(S(t, u)) d\mu_0(u)$ is a continuous function of $t \in (0, T)$.

2.4. The notion of a statistical solution is a certain generalisation of the usual one in the sense described by the following theorem.

THEOREM 5. *A function $u: [0, T] \rightarrow H_0^1$ satisfies (2) for almost all $\tau \in [0, T]$ and all v such as in sec. 1.5. if and only if a family of Dirac measures $\{\delta_{u(t)}\}_{t \in (0, T)}$ is a statistical solution of the heat equation with the initial measure $\delta_{u_0}(u_0 = u(0))$.*

Proof. If u satisfies (2) then

$$\delta_{u(t)}(\omega) = \delta_{u_0}(S(t, \cdot)^{-1}(\omega))$$

so that by means of theorem 4 implication from the left to the right is obvious.

Conversely let us assume that $\{\delta_{u(t)}\}$ is a statistical solution. For any $\varphi \in \mathcal{F}^{\text{ind}}$ we have

$$\varphi(u(\tau)) - \varphi(u_0) + \int_0^\tau (u(t), \varphi'(u(t)))_1 dt = \int_0^\tau (f(t), \varphi'(u(t))) dt \quad \text{a.e. on } (0, T).$$

We take $\varphi(u) := (u, v)$ where v is arbitrarily fixed in H_0^1 . It is easy to see that the functional so defined belongs to \mathcal{F}^{ind} ($\varphi'(u) = v$), hence

$$(u(\tau), v) - (u_0, v) + \int_0^\tau (u(t), v)_1 dt = \int_0^\tau (f(t), v) dt.$$

We shall prove that u satisfies (2) with test functions of the form $g(t) \cdot v$ where $g: [0, T] \rightarrow \mathbb{R}$ is differentiable, g' is integrable and $\text{supp } g \subset [0, T]$.

$$\begin{aligned}
 & (u(\tau), g(\tau)v) - (u_0, g(0)v) - \int_0^\tau (u(t), g'(t)v) dt + \\
 & \quad + \int_0^\tau [(u(t), g(t)v)_1 - (f(t), g(t)v)] dt \\
 & = (u(\tau), g(\tau)v) - (u_0, g(0)v) - \int_0^\tau (u(t), g'(t)v) dt + \\
 & \quad + g(\tau) \int_0^\tau [(u(t), v)_1 - (f(t), v)] dt - \\
 & \quad - \int_0^\tau g'(t) \int_0^t [(u(s), v)_1 - (f(s), v)] ds dt \\
 & = (u(\tau), g(\tau)v) - (u_0, g(0)v) - \int_0^\tau (u(t), g'(t)v) dt + \\
 & \quad + (u_0, g(\tau)v) - (u(\tau), g(\tau)v) - \int_0^\tau g'(t)(u_0, v) dt + \\
 & \quad + \int_0^\tau (u(t), g'(t)v) dt \\
 & = [g(\tau) - g(0)](u_0, v) - (u_0, v) \int_0^\tau g'(t) dt = 0 \quad \text{a.e. on } (0, T).
 \end{aligned}$$

Now we take v as in sec. 1.5., i.e. $v \in C(0, T; H_0^1)$ with $v'_i \in L^2(0, T; L^2)$ and $\text{supp } v \subset [0, T]$.

$$\begin{aligned}
 v(t) &= \sum_{i=1}^{\infty} (v(t), w_i) w_i, \\
 v'_i(t) &= \sum_{i=1}^{\infty} (v'_i(t), w_i) w_i.
 \end{aligned}$$

We put $g_i(t) := (v(t), w_i)$ ($i = 1, 2, \dots$). g_i are differentiable and $g'_i(t) = (v'_i(t), w_i)$. g'_i are integrable in virtue of $|(v'_i(t), w_i)| \leq \|v'_i(t)\|$, thus (2) holds for test functions of the form

$$G_n(t) := \sum_{i=1}^n g_i(t) w_i.$$

$G_n(t) \rightarrow v(t)$, $G'_n(t) \rightarrow v'_i(t)$ as $n \rightarrow \infty$ (convergence in L^2) so that

$$\begin{aligned}
 (u(\tau), G_n(\tau)) &\rightarrow (u(\tau), v(\tau)), \\
 (u_0, G_n(0)) &\rightarrow (u_0, v(0)).
 \end{aligned}$$

$$\begin{aligned}
 |(u(t), G'_n(t))| &\leq \sum_{i=1}^n |(v'_i(t), w_i)| \cdot |(u(t), w_i)| \\
 &\leq \sum_{i=1}^n (|(v'_i(t), w_i)|^2 + |(u(t), w_i)|^2) \\
 &\leq \|v'_i(t)\|^2 + \|u(t)\|^2.
 \end{aligned}$$

$\{\delta_{u(t)}\}_{t \in (0, T)}$ is a basic family of measures, hence the majorant of the sequence $\{(u(t), G'_n(t))\}$ is t integrable on $(0, T)$.

Applying the Lebesgue's theorem on dominated convergence we get

$$\int_0^\tau (u(t), G'_n(t)) dt \rightarrow \int_0^\tau (u(t), v'_i(t)) dt \quad \text{as } n \rightarrow \infty.$$

We have

$$\begin{aligned} |(u(t), G_n(t))_1| &\leq \sum_{i=1}^n |(u(t), w_i)_1| \cdot |(v(t), w_i)| \\ &\leq \sum_{i=1}^n \lambda_i |(u(t), w_i)| \cdot |(v(t), w_i)| \\ &\leq \sum_{i=1}^n \lambda_i |(u(t), w_i)|^2 + \sum_{i=1}^n \lambda_i |(v(t), w_i)|^2 \\ &\leq \|u(t)\|_1^2 + \|v(t)\|_1^2, \end{aligned}$$

and similarly

$$|(f(t), G_n(t))| \leq \|f(t)\|^2 + \|v(t)\|^2.$$

Thus applying Lebesgue's theorem once more we get

$$\int_0^\tau [(u(t), G_n(t))_1 - (f(t), G_n(t))] dt \rightarrow \int_0^\tau [(u(t), v(t))_1 - (f(t), v(t))] dt \quad \text{as } n \rightarrow \infty,$$

theorem is proved.

Remark. We can prove in the same way the following theorem:

THEOREM 5'. $u: [0, T) \rightarrow H_0^1$ is a solution of the heat equation with initial data u_0 in the sense of theorem 1 ((2) holds for all $\tau \in [0, T)$) if and only if $\{\delta_{u(t)}\}_{t \in (0, T)}$ is a statistical solution of the heat equation with initial measure δ_{u_0} in the sense of definition 4'.

3. The problem of uniqueness for statistical solutions

3.1. In this section we shall prove a theorem showing equivalent forms of equation (7).

Definition 5. $\varphi: [0, T) \times L^2 \rightarrow R$ will be called *test functional* ($\varphi \in \mathcal{F}$) if

- 1) φ is continuous,
- 2) for any fixed $u \in L^2$, $\varphi(\cdot, u)$ is of the class C^1 and

$$|\varphi'_1(t, u)| \leq c_{16} + c_{17} \|u\| \quad \text{for all } t, u,$$

where $\varphi'_1(t, u) = \frac{\partial}{\partial t} \varphi(t, u)$.

- 3) for any fixed $t \in [0, T)$, for each $u \in L^2$ there exists $\varphi'_2(t, u) \in H_0^1$ such that

$$\frac{|\varphi(t, u+v) - \varphi(t, u) - (\varphi'_2(t, u), v)|}{\|v\|} \rightarrow 0$$

as $\|v\| \rightarrow 0$, $v \in H_0^1$. Moreover $\varphi'_2(\cdot, \cdot): [0, T] \times L^2 \rightarrow H_2^1$ is continuous and bounded: $\|\varphi'_2(t, u)\|_1 \leq c_{18}$.

Definition 6. $\varphi \in \mathcal{F}$ will be called *elementary* ($\varphi \in \mathcal{F}_0$) if for some m (depending on φ) we have $\varphi(t, u) = \varphi(t, P_m u)$ for $t \in [0, T]$, $u \in L^2$.

LEMMA 2. For any $\varphi \in \mathcal{F}$, $t \in [0, T]$ there exist constants c_{19} , c_{20} such, that

$$|\varphi(t, u)| \leq c_{19} + c_{20} \|u\| \quad \text{for all } u \in L^2.$$

Proof. For any fixed $\varphi \in \mathcal{F}$, $u, v \in L^2$, $s \in [0, 1]$ we put

$$g(s) := \varphi(t, su + (1-s)v).$$

Obviously

$$g'(s) = (\varphi'_2(t, su + (1-s)v), u - v).$$

$$\begin{aligned} |\varphi(t, u) - \varphi(t, v)| &= \left| \int_0^1 (\varphi'_2(t, su + (1-s)v), u - v) ds \right| \\ &\leq \frac{\|u - v, \varphi'_2(t, \bar{s}u + (1-\bar{s})v)\|}{\|\varphi'_2(t, \bar{s}u + (1-\bar{s})v)\|_1} \|\varphi'_2(t, \bar{s}u + (1-\bar{s})v)\|_1 \\ &\leq \|u - v\|_{-1} c_{18} \quad (s \in [0, 1]). \end{aligned}$$

Now we find constants c_{19} , c_{20} fixing v , thus lemma 2 is proved.

LEMMA 3. φ belongs to \mathcal{F}^{ind} if and only if for each $r \in C_0^\infty([0, \infty))$ $\psi(t, u) = r(t)\varphi(u)$ belongs to \mathcal{F} .

Proof. First we assume $\varphi \in \mathcal{F}$.

$$|\psi'_1(t, u)| = |r'(t)\varphi(u)| \leq \max |r'(t)| (c_9 + c_{10} \|u\|).$$

$$\frac{|\psi(t, u+v) - \psi(t, u) - (r(t)\varphi'(u), v)|}{\|v\|} = r(t) \frac{|\varphi(u+v) - \varphi(u) - (\varphi'(u), v)|}{\|v\|} \rightarrow 0$$

as $\|v\| \rightarrow 0$, $v \in H_0^1$, hence $\psi'_2(t, u) = r(t)\varphi'(u)$. Now the implication from the left to the right is obvious. Conversely we take $r \in C_0^\infty([0, \infty))$ not vanishing identically. We have

$$|\varphi(u)r'(t)| \leq c_{16} + c_{17} \|u\|.$$

Fixing t so that $r'(t) \neq 0$ we get

$$|\varphi(u)| \leq \frac{c_{16}}{|r'(t)|} + \frac{c_{17}}{|r'(t)|} \|u\| \quad \text{for all } u \in L^2.$$

It is easy to verify that $\varphi'(u) = \frac{1}{r(t)} \psi'_2(t, u)$ (t is fixed) and the remaining conditions are obvious; thus lemma 3 is proved.

THEOREM 6. For a basic family of measures $\{\mu_t\}_{t \in (0, T)}$ the three following conditions are equivalent:

(i) $\{\mu_t\}_{t \in (0, T)}$ is a statistical solution of the heat equation with the initial measure μ_0 , i. e.

$$(7) \quad \int_{H_0^1} \varphi(u) d\mu_\tau(u) - \int_{L^2} \varphi(u) d\mu_0(u) + \int_0^\tau \int_{H_0^1} (u, \varphi'(u))_1 d\mu_t(u) dt = \int_0^\tau \int_{H_0^1} (f(t), \varphi'(u)) d\mu_t(u) dt$$

holds for all $\varphi \in \mathcal{F}^{\text{ind}}$ and all $\tau \in B \subset (0, T)$:

(ii)

$$(14) \quad \int_{H_0^1} \psi(\tau, u) d\mu_\tau(u) - \int_{L^2} \psi(0, u) d\mu_0(u) + \int_0^\tau \int_{H_0^1} [-\psi'_1(t, u) + (u, \psi'_2(t, u))_1] d\mu_t(u) dt = \int_0^\tau \int_{H_0^1} (f(t), \psi'_2(t, u)) d\mu_t(u) dt$$

holds for all $\psi \in \mathcal{F}$ and all $\tau \in B \subset (0, T)$;

(iii) (14) holds for all ψ of the form $\psi(t, u) = r(t)\varphi(u)$ where $r \in C_0^\infty([0, \infty))$, $\varphi \in \mathcal{F}^{\text{ind}}$; and all $\tau \in B \subset (0, T)$; where B is a set of full measure in $(0, T)$ and does not depend on the choice of test functional.

Proof. (i) \Rightarrow (iii). We take any $\varphi \in \mathcal{F}^{\text{ind}}$, $r \in C_0^\infty([0, \infty))$ and we put

$$G_\varphi(\tau) := \int_0^\tau \int_{H_0^1} [(f(t), \varphi'(u)) - (u, \varphi'(u))_1] d\mu_t(u) dt + \int_{L^2} \varphi(u) d\mu_0(u) \quad \text{for } \tau \in [0, T).$$

G_φ is an absolutely continuous function and for almost all $t \in (0, T)$ $G_\varphi(\tau) = \int_{H_0^1} \varphi(u) d\mu_t(u)$.

$$\begin{aligned} & \int_0^\tau \int_{H_0^1} [-r'(t)\varphi(u) + r(t)(u, \varphi'(u))_1 - r(t)(f(t), \varphi'(u))] d\mu_t(u) dt \\ &= r(\tau) \int_{H_0^1} \varphi(u) d\mu_\tau(u) - r(0) \int_{L^2} \varphi(u) d\mu_0(u) \\ & \quad + \int_0^\tau r(t) \left(\frac{d}{dt} G_\varphi(t) + \int_{H_0^1} [(u, \varphi'(u))_1 - (f(t), \varphi'(u))] d\mu_t(u) \right) dt \end{aligned}$$

for all $\tau \in B$. From (7) we get

$$\frac{d}{dt} G_\varphi(t) = \int_{H_0^1} [(f(t), \varphi'(u)) - (u, \varphi'(u))_1] d\mu_t(u)$$

