

## A statistical approach to the heat equation

by M. CAPIŃSKI

In [1] C. Foias has introduced the notion of a statistical solution of the Navier-Stokes equations. In this paper we apply his methods to the heat equation, giving detailed proofs of the existence (Th. 4) and global uniqueness (Th. 7) theorems.

### 1. Introduction

**1.1.** Let  $\Omega$  be a bounded domain in  $R^n$  with boundary of class  $C^2$ . The Hilbert space of measurable real valued functions  $u$  defined on  $\Omega$  with a finite norm

$$\|u\| := \left( \int_{\Omega} |u(x)|^2 dx \right)^{1/2}$$

will be denoted by  $L^2$ . We put

$$\int_{\Omega} u(x)v(x)dx =: (u, v) \quad \text{for } u, v \in L^2.$$

We define the subspace  $H_0^1$  of  $L^2$  in the following way:  $H_0^1$  is the closure of  $C_0^\infty(\Omega)$  in  $H^1$

where  $H^1 = \left\{ u \in L^2 : \frac{\partial u}{\partial x_k} \in L^2 (k = 1, \dots, n) \right\}$  (derivatives are taken in the sense of the

theory of distributions) with norm  $\|u\|_{H^1} = \left( \|u\|^2 + \sum_{k=1}^n \left\| \frac{\partial u}{\partial x_k} \right\|^2 \right)^{1/2}$ .  $H_0^1$  is a Hilbert

space with a scalar product

$$(u, v)_1 := \sum_{k=1}^n \left( \frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k} \right) \quad \text{for } u, v \in H_0^1.$$

We denote  $\|u\|_1 := ((u, u)_1)^{1/2}$  for  $u \in H_0^1$ .

1.2. Let us consider the operator  $-\Delta$  defined on  $C_0^\infty(\Omega)$ . There is a unique self-adjoint extension  $A$  of  $-\Delta$  such that

$$\begin{aligned} D(A) &\subset H_0^1, \\ D(A^{1/2}) &= H_0^1, \\ (u, v)_1 &= (A^{1/2}u, A^{1/2}v), \\ \|u\| &\leq c_1 \|u\|_1 \end{aligned}$$

for  $u, v \in H_0^1$  (see [2] p. 191, [3] p. 110, [4] p. 24).

1.3. For any fixed  $u \in L^2$  applying the Riesz theorem for a functional  $(u, \cdot): H_0^1 \rightarrow \mathbb{R}$  we get a unique  $Iu \in H_0^1$  satisfying

$$(u, v) = (Iu, v)_1 \quad \text{for } v \in H_0^1.$$

It is easy to see that  $I: L^2 \rightarrow H_0^1$  is linear, continuous and such that  $\|Iu\|_1 \leq c_1 \|u\|$  for  $u \in L^2$ . The complete extension of the unitary space  $(L^2, (\cdot, \cdot)_{-1})$  where

$$(u, v)_{-1} := (Iu, v) = (Iu, Iv)_1$$

will be denoted by  $H_0^{-1}$ . We have

$$\begin{aligned} H_0^1 &\subset L^2 \subset H_0^{-1}, \\ \|u\|_{-1} &\leq c_1 \|u\| \quad \text{for } u \in L^2. \end{aligned}$$

Extending continuously  $I$  on  $H_0^{-1}$  we get the bijection  $I: H_0^{-1} \rightarrow H_0^1$  such that

$$(\alpha, \beta)_{-1} = (I\alpha, I\beta)_1 \quad \text{for } \alpha, \beta \in H_0^{-1}.$$

For  $\alpha \in H_0^{-1}$ ,  $u \in H_0^1$  we put

$$\alpha[u] := \lim_{n \rightarrow \infty} (v_n, u)$$

where  $v_n \rightarrow \alpha$ ,  $v_n \in H_0^1$ . We have the following properties

$$\begin{aligned} |\alpha[u]| &\leq \|\alpha\|_{-1} \|u\|_1, \\ \alpha[u] &= (I\alpha, u)_1. \end{aligned}$$

$H_0^{-1}$  can be identified in the following sense with a set of all continuous linear functionals on  $H_0^1$ . For  $\alpha \in H_0^{-1}$   $l_\alpha(u) := \alpha[u]$  is a linear continuous functional  $l_\alpha: H_0^1 \rightarrow \mathbb{R}$ ,  $\|\alpha\|_{-1} = \|l_\alpha\| := \sup_{u \in H_0^1} \frac{|l_\alpha(u)|}{\|u\|_1}$ . Conversely for  $l: H_0^1 \rightarrow \mathbb{R}$  linear and continuous there exists a unique  $v_l \in H_0^1$  such that  $l(u) = (v_l, u)_1 = I^{-1}v_l[u]$ . We put  $\alpha_l := I^{-1}v_l \in H_0^{-1}$  obtaining  $\|l\| = \|v_l\|_1 = \|\alpha_l\|_{-1}$ . (for details see [5] ch. I).

1.4. We extend the operator  $A$  on the whole  $H_0^1$ . For  $u \in H_0^1$  we put

$$A_e u := I^{-1}u \in H_0^{-1}.$$

We have

$$A_e u[v] = (u, v)_1 \quad \text{for any } v \in H_0^1.$$

For  $u \in D(A)$ ,  $v \in H_0^1$  we get

$$(A_\epsilon u, v)_{-1} = (IA_\epsilon u, Iv)_1 = (u, Iv)_1 = (A^{1/2}u, A^{1/2}Iv) = (Au, Iv) = (Au, v)_{-1}$$

hence  $A_\epsilon u = Au$ .

**1.5.** In [6] (p. 163) the following theorem may be found:

**THEOREM 1.** Let us fix any  $f \in L^2(0, T; L^2)$  where  $T > 0$  or  $T = +\infty$ . For each  $u_0 \in L^2$  there exists exactly one  $u: [0, T) \rightarrow L^2$  such that

- 1)  $u(t) \in H_0^1$  for  $t \in (0, T)$ ,
- 2) there exists a constant  $c_2$  such that

$$(1) \quad \text{ess sup}_{t \in [0, T)} \|u(t)\| + \left( \int_0^T \|u(t)\|_1^2 dt \right)^{1/2} \leq c_2 \left( \frac{1}{2} \|u_0\| + \int_0^T \|f(t)\| dt \right),$$

$$(2) \quad 3) \quad (u(\tau), v(\tau)) - (u_0, v(0)) - \int_0^\tau (u(t), v'_t(t)) dt + \int_0^\tau (u(t), v(t))_1 dt = \int_0^\tau (f(t), v(t)) dt$$

holds for all  $\tau \in [0, T)$ ,  $v \in C(0, T; H_0^1)$  such that there exists  $v'_t \in L^2(0, T; L^2)$ ,

4) the mapping  $S: [0, T) \times L^2 \rightarrow L^2$  defined as follows  $S(t, u_0) := u(t)$  is continuous with respect to each variable.

**1.6.** We shall prove the following corollary of Th. 1.

**THEOREM 2.** Let  $f \in L^2(0, T; L^2)$ . For every  $u_0 \in L^2$  there exists exactly one  $u: [0, T) \rightarrow L^2$  such that

- 1)  $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1)$ ,
- 2)  $u$  has a weak derivative  $u'$  (for definition see [7] p. 387) as a function  $u: [0, T) \rightarrow H_0^{-1}$ ,
- 3)  $\begin{cases} u'(t) + A_\epsilon u(t) = f(t) & \text{for almost all } t \in (0, T) \\ u(0) = u_0, \end{cases}$
- 4) there exist constants  $c_3, \dots, c_6$  such that

$$(3) \quad \begin{cases} \|u(t)\|^2 \leq c_3 \|u_0\|^2 + c_4 & \text{a.e. on } (0, T) \\ \int_0^T \|u(t)\|_1^2 dt \leq c_5 \|u_0\|^2 + c_6. \end{cases}$$

**Proof:** We take  $u$  obtained in Th. 1. For every  $v \in C_0(0, T; H_0^{-1})$  having integrable  $v'_t: (0, T) \rightarrow H_0^{-1}$  we have

$$-\int_0^T (u(t), v'_t(t))_{-1} dt = -\int_0^T (u(t), I(v'_t(t))) dt.$$

It is easy to see that  $(I \circ v)'$  exists and that  $(I \circ v)' = I \circ v'$ . This yields

$$\begin{aligned} -\int_0^T (u(t), I(v'_t(t))) dt &= -\int_0^T (u(t), (I \circ v)'(t)) dt \\ &= \int_0^T [-(u(t), I \circ v(t))_1 + (f(t), I \circ v(t))] dt \end{aligned}$$

