ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DII PRACE MATEMATYCZNE, Z. 20 1979

Stability and Lyapunov functions in generalized local pseudo-dynamical systems

by A. TRZEPIZUR

Introduction. The purpose of the present paper is to give a common scheme for the different types of stability. D. Bushaw ([3]) formulated a general concept of stability of the Lyapunov type for a flow on the metric space. This idea is used in the present paper in the construction of a general formula of stability in generalized local pseudo-dynamical systems. The method of construction of the conditions for the Lyapunov function is different from Dana's method in paper [4]. It is some further generalization of A. Pelczar's conception ([6], [7], [8]). In Sections 1 and 2 we give a definition of a generalized local pseudo-dynamical system and a general definition of the stability of a set. Section 3 contains the main theorems of the paper. These theorems (2 and 3) give a construction of the Lyapunov function for many types of stability. We give some applications of Theorems 2 and 3 in Section 4.

1. We shall use the notation and definitions from [3], [6], [7], [8].

Let X be a non-empty set, $(U, +)$ be an abelian semi-group having the neutral element 0 and let $A \subset U \times X$ be a set such that

$$\{0\} \times X \subset A .$$

Then we put for any $x \in X$

$$I_x = \{t \in U\colon (t, x) \in A\} .$$

Let $\mathscr{P}(X)$ denote the family of non-empty subsets of X and let

$$\lambda\colon A \to \mathscr{P}(X)$$

be a mapping.

Definition 1. A quadruplet (X, U, A, λ) is said to be a *generalized local pseudo-dynamical system* if and only if the following three conditions hold:

(i) $\lambda(0, x) = \{x\}$ for every $x \in X$,

(ii) $s+t \in I_x$ for every $s \in I_x$, $y \in \lambda(s, x)$, $t \in I_y$,

(iii) $\lambda(t, y) \subset \lambda(t+s, x)$ for every $s \in I_x$, $y \in \lambda(s, x)$, $t \in I_y$.

If $A = U \times X$ (or $I_x = U$ for every $x \in X$), (X, U, A, λ) is called the *generalized global pseudo-dynamical system* (or in brief: the *generalized pseudo-dynamical system*). It is obvious that the generalized local pseudo-dynamical system is the generalization of the local pseudo-dynamical system.

Let (X, U, A, λ) be a generalized local pseudo-dynamical system. We put

$$\lambda_t(x) = \lambda(t, x)\colon\ (t, x) \in A$$

$$\lambda(x) = \bigcup \{\lambda_t(x)\colon\ t \in I_x\}\colon\ x \in X$$

$$\lambda(B) = \bigcup \{\lambda(x)\colon\ x \in B\}\colon\ B \subset X.$$

The set $\lambda(x)$ is called the *trajectory* of the point x.

Definition 2. We say that a set $M \subset X$ is *invariant* if and only if $\lambda(x) \subset M$ for every $x \in M$.

2. We suppose that the generalized local pseudo-dynamical system (X, U, A, λ) is given. Let $\beta\colon X \to \mathscr{P}(\mathscr{P}(X))$ be a mapping on X, $\Omega \subset \mathscr{P}(X)$, $\Omega \neq \emptyset$ and let $M \in \mathscr{P}(X)$.

Remark 1. If X is a metric space and $M \in \mathscr{P}(X)$, then we may put:

$$\beta\colon\ X \ni x \to \{K(x, \varepsilon)\}_{\varepsilon > 0} \in \mathscr{P}(\mathscr{P}(X))$$

and

$$\Omega = \{K(M, \varepsilon)\}_{\varepsilon > 0},$$

where $K(B, \varepsilon) = \{y \in X\colon\ \varrho(y, B) < \varepsilon\}$, $K(x, \varepsilon) = K(\{x\}, \varepsilon)$ for every $x \in X$ and $B \subset X$.

Next we put $\tilde{\beta} = \bigcup \{\beta(x)\colon\ x \in X\}$. The following table contains the variables, their domains and the quantifiers. We shall denote each variable by a Greek letter and the corresponding universal or existential quantifier by the corresponding roman upper-case or lower-case letter, respectively

variable	domain	existential quantifier	universal quantifier
ω	Ω	o	O
ζ	M	z	Z
δ	$\tilde{\beta}$	d	D
η	X	h	H
ν	U	n	N

For example: o denotes $\exists \omega \in \Omega$,
D denotes $\forall \delta \in \tilde{\beta}$.

The expression

$$\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda(\lambda_\nu(\eta)) \subset \omega)],$$

which will be denoted by Φ, is a *propositional function* in the five variables ω, ζ, δ, η and ν (the symbols $\underset{\delta}{\supset}$, $\underset{\eta}{\supset}$, $\underset{\nu}{\supset}$ will be explained later). Any sequence of letters from the third and fourth columns of table we shall call a *word*. If the word contains only one letter from each row, then we say that the word is *feasible*. The class of feasible words will be denoted by $\mathscr{S}$.

Let $S \in \mathscr{S}$. Let us replace the letters of S by the quantifiers. If we put this sequence of five quantifiers before the formula Φ we shall obtain a formula without the free variables, which will be denoted by $S\Phi$. Let ι stand for any of the variables in the table. Then the symbol $\underset{\iota}{\supset}$ in the formula $S\Phi$ denotes $\wedge$ (conjunction) if it appears within the scope of the corresponding existential quantifier and denotes $\Rightarrow$ (implication) otherwise.

Definition 3. We say that a set M is (S, Ω, β)-*stable* if and only if the formula $S\Phi$ is true.

This definition contains many types of stability. We shall give some examples. In these examples X is a metric space, (X, R_*, A, λ) ($R_* = \{t \in R\colon t \geqslant 0\}$) is a generalized local pseudo-dynamical system, $M \in \mathscr{P}(X)$ and $\Omega = \{K(M, r)\}_{r>0}$. In examples 1 and 2 we put

$$\beta\colon\ X \ni x \to \{K(x, r)\}_{r>0} \in \mathscr{P}(\mathscr{P}(X))$$

in example 3

$$\beta\colon\ X \ni x \to \{K(M, r)\}_{r>0} \in \mathscr{P}(\mathscr{P}(X)) .$$

Example 1. A set M is Lyapunov stable (the classical definition), if and only if for every $x \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda(y) \subset K(M, \varepsilon)$ for every $y \in K(x, \delta)$. This stability is equivalent to the (ZOdHN, Ω, β)-stability of the set M.

Example 2. Let us consider the weak stability of set M. A set M is *weakly stable* if and only if for every $x \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in K(x, \delta)$, $\lambda(\lambda_s(y)) \subset K(M, \varepsilon)$ for the same $s \in I_y$. This is the (ZOdHn, Ω, β)-stability of M.

Example 3. We suppose that the mapping $\lambda = \pi$ gives the global pseudo-dynamical system $(X, R_*, R_* \times X, \pi)$. We say that M is an *attractor* if and only if there exists $\delta > 0$ such that $\lim_{t \to \infty} \varrho(\pi(t, y), M) = 0$ for every $y \in K(M, \delta)$. It is obvious that this condition gives the (ZdHOn, Ω, β)-stability of M.

Remark 2. If we put the quantifier $\exists s \in R_*$ in different places in example 2 we shall obtain a large number of interesting variants of Lyapunov stability. Connections between these variants for pseudo-dynamical systems are presented in paper [7].

3. The definition of stability given above contains many types of stability. It is obvious that in the case of classical Lyapunov stability there exists a Lyapunov function of some properties. We shall generalize this well-known case for every word S and Ω, β. In order to find the conditions of a Lyapunov function we shall construct a certain family of sets $\subset \mathscr{P}(X)$, the existence of which is equivalent to the (S, Ω, β)-stability considered. This is a generalization of A. Pelczar's idea given in paper [6] for classical Lyapunov stability.

Let (X, U, A, λ) be a generalized local pseudo-dynamical system and let S, Ω, β be defined as above. We shall add one variable α the domain of which will be $\mathscr{A} \subset \mathscr{P}(X)$, $\mathscr{A} \neq \emptyset$. We shall denote the corresponding quantifiers a, A in accordance with convention.

Let us contract two words S_1 and S_2 associated with S. In order to make the word S_1 we put (a, A) in the word S instead of (o, O). The word S_2 is denoted in the following manner:

$$S_2 = \begin{cases} Oa, & \text{if} \quad O \text{ occurs in the word } S \\ Ao, & \text{if} \quad o \text{ occurs in the word } S. \end{cases}$$

Let us take formulas P_1, P_2, P_3

$$P_1: \ \delta \in \beta(\zeta) \underset{\delta}{\Rightarrow} [\eta \in \delta \underset{\eta}{\Rightarrow} (\nu \in I_\eta \underset{\nu}{\Rightarrow} \lambda_\nu(\eta) \subset \alpha)],$$

$$P_2: \ \alpha \subset \omega,$$

$$P_3: \ \forall \alpha \in \mathscr{A}, \ \alpha \text{ is an invariant set.}$$

The following theorem gives the condition equivalent to the considered (S, Ω, β)-stability of M (this is a generalization of condition $L(\Omega)$ given in [6]).

THEOREM 1. *Let (X, U, A, λ) be a generalized local pseudo-dynamical system, $\Omega \neq \emptyset$, $\Omega \subset \mathscr{P}(X)$, β: $X \to \mathscr{P}(\mathscr{P}(X))$ and $M \in \mathscr{P}(X)$. Then for every word $S \in \mathscr{S}$ the following equivalence is true: M is (S, Ω, β)-stable if and only if there exists a non-empty subfamily $\mathscr{A} \subset \mathscr{P}(X)$ such that*

(a) $S_1 P_1$,

(b) $S_2 P_2$,

(c) P_3.

First shall prove the following lemma.

LEMMA 1. *Let us suppose that there exists a family $\mathscr{C} \subset \mathscr{P}(X) \cup \{\emptyset\}$ $(= \{Y\colon Y \subset X\})$ which fulfils the conditions* (a), (b), (c) *of the theorem (it is obvious that the domain of α is $\mathscr{C}$). Then there exists a family $\mathscr{A} \subset \mathscr{P}(X)$, $\mathscr{A} \neq \emptyset$, which fulfils the same conditions.*

Proof. Let us assume that O occurs in the word S. As a result of the assumptions of the lemma the domain of letters ζ, δ, η, ν is non-empty for every ζ, δ, η, ν of the corresponding domain. Since $\mathscr{C}$ fulfils the condition (a) thus $\alpha \neq \emptyset$ for every $\alpha \in \mathscr{C}$ which ever quantifiers are in front of variables ζ, δ, η, ν in the word S. In virtue of (b) the family $\mathscr{C}$ is non-empty. We put $\mathscr{A} = \mathscr{C}$ in this case.

Now we suppose that o occurs in the word S. In virtue of (a) we obtain that there exists a set $\alpha \in \mathscr{C}$, $\alpha \neq \emptyset$. Thus the family $\mathscr{A} = \{\alpha \in \mathscr{C}\colon \alpha \neq \emptyset\}$ is non-empty, $\subset \mathscr{P}(X)$. It is obvious that $\mathscr{A}$ fulfils (a), (b), (c) as well.

Proof of Theorem 1. We suppose that there exists a family $\mathscr{A} \subset \mathscr{P}(X)$, $\mathscr{A} \neq \emptyset$ which fulfils the conditions (a)—(c). Let O occurs in the word S. Since the quantifiers in front of O are the same in the words S and S_1, we make the same corresponding choices up to the position of O in the formula $S\Phi$ as in the condition $S_1 P_1$. Next we find an arbitrary $\omega \in \Omega$. We choose $\alpha \in \mathscr{A}$ such that $\alpha \subset \omega$ (in virtue of (b)). We finish the condition $S_1 P_1$ with this α and we obtain the whole choice in the formula of (S, Ω, β)-stability. Now it only remains to check that the choice is correct. Let us consider formula P_1:

$$\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda_\nu(\eta) \subset \alpha)] .$$

We may write the last term in the following form

$$\forall y \in \lambda_\nu(\eta),\, y \in \alpha .$$

Since the set α is invariant (from condition (c)) we have

$$\forall y \in \lambda_\nu(\eta), \quad \lambda(y) \subset \alpha ,$$

i.e.

$$\lambda(\lambda_\nu(\eta)) \subset \alpha .$$

As $\alpha \subset \omega$, we have

$$S\{\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda(\lambda_\nu(\eta)) \subset \omega)]\} ,$$

i.e. $S\Phi$ under the previous choice, which proves the proof in the case of O in the word S. Let us suppose now that o occurs in the word S. Since the quantifiers, up to the position of o, are the same in the words S and S_1, we make corresponding choices, up to the position of o, the same as in $S_1 P_1$. Coming back to the word S we find that we have to shaw a certain $\omega \in \Omega$. We choose it for $\alpha \in \mathscr{A}$ obtained from the formula $S_1 P_1$ from the condition (b). We make the same final choice in the formula $S\Phi$ as in the $S_1 P_1$. Further argumentation is the same as in the previous case, and next we obtain the proof of the fact that the existence of a family $\mathscr{A}$ fulfilling (a)—(c) is sufficient for the (S, Ω, β)-stability of M.

Now let us assume that M is (S, Ω, β)-stable. We define the family $\mathscr{C}$ in the following manner

$$\mathscr{C} = \{E(\omega) \colon\ \omega \in \Omega\}$$

where $E(\omega) = \{y \in X \colon\ \lambda(y) \subset \omega\}$. If we show that $\mathscr{C}$ fulfils (a), (b), (c), then the proof of Theorem 1 will be completed (in virtue of Lemma 1).

We have

$$S\{\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda(\lambda_\nu(\eta)) \subset \omega)]\} .$$

The last term we may write in the form

$$\forall y \in \lambda_\nu(\eta), \quad y \in E(\omega)$$

or

$$\lambda_\nu(\eta) \subset E(\omega) .$$

Now it is sufficient to put the corresponding quantifier of letter ω. Then we obtain

$$S_1\{\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda_\nu(\eta) \subset \alpha)]\} .$$

Thus the family $\mathscr{C}$ fulfils the condition (a).

If we show that $E(\omega) \subset \omega$ for every $\omega \in \Omega$, then (b) will be satisfied. Let $y \in E(\omega)$ be fixed. Then $\lambda(y) \subset \omega$, and $y \in \omega$. Now we shall that $\mathscr{C}$ fulfils the condition (c). Let $\alpha \in \mathscr{C}$, where $\alpha = E(\omega)$ for the same $\omega \in \Omega$, be fixed. Let $y \in E(\omega)$, $s \in I_y$, $z \in \lambda_s(y)$ and $t \in I_z$. Hence we have

$$\lambda_t(z) \subset \lambda_{t+s}(y)$$

in virtue of definition 1. Since $y \in E(\omega)$ we obtain $\lambda_t(z) \subset \omega$. Thus $\lambda_s(y) \subset E(\omega) = \alpha$. The set α is invariant. The proof of Theorem 1 is completed.

In the formula of the Theorem 1 the function λ occurs in the conditions (a), (c). In the condition (a) we may omit λ in certain feasible words. For example the feasible words in which N is in the last position in the word belong to this group. The class of these words we denote by $\overline{\mathcal{S}}$.

Let $S \in \overline{\mathcal{S}}$ be arbitrarily fixed. Then we may write the condition of the (S, Ω, β)-stability of the set M in the simple form

$$\bar{S}\{\delta \in \beta(\zeta) \underset{\delta}{\supset} (\eta \in \delta \underset{\eta}{\supset} \lambda(\eta) \subset \omega)\},$$

where the symbol "$-$" denotes the omission of the letter N.

Theorem 1 gives the following

COROLLARY 1. *If $S \in \overline{\mathcal{S}}$, then a set M is (S, Ω, β)-stable if and only if there exists a family $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{A} \neq \emptyset$ such that*

$$(\bar{a}) \quad \bar{S}_1\{\delta \in \beta(\zeta) \underset{\delta}{\supset} (\eta \in \delta \underset{\eta}{\supset} \eta \in \alpha)\}$$

$$(\bar{b}) \quad S_2\{\alpha \subset \omega\}$$

$$(\ddot{c}) \quad P_3 .$$

It is obvious that the function λ does not occur in the condition (a).

Let $(T, \leqslant)$ be a partially ordered space. We shall denote

$$P = \begin{cases} T, & \text{if } \inf T \text{ does not exist} \\ T \setminus \{\inf T\}, & \text{if there exist } \inf T . \end{cases}$$

Let (X, U, A, λ) be a general local pseudo-dynamical system and let S, M, Ω, β be as above.

Definition 4. *A* function $V\colon W \to T$ where $\emptyset \neq W \subset X$ is called the *(S, Ω, β)-Lyapunov function for the set M* if and only if the family $\mathcal{A} = \{A_\xi\}_{\xi \in P}$, where $A_\xi = \{y \in W\colon V(y) \leqslant \xi\}$, fulfils the condition (a) and (b) from Theorem 1 and the condition

$$(c') \quad \forall y \in W \forall z \in \lambda(y),\ z \in W \wedge V(z) \leqslant V(y) .$$

We can see immediately from the definition that W is invariant.

Remark 3. If $S \in \overline{\mathcal{S}}$, we can show that the function $V\colon W \to T$ is a (S, Ω, β)-Lyapunov function for M if and only if the family $\mathcal{A}$ fulfils the conditions (a) and (b) from Corollary 1 and the condition (c′) from the previous definition.

Now we shall give the fundamental theorems of this paper.

THEOREM 2. *If there exists the (S, Ω, β)-Lyapunov function for the set M, then M is (S, Ω, β)-stable.*

Proof. If we show that $\mathscr{A} = \{A_\xi\}_{\xi \in P}$ fulfils condition (c), i.e.

$$\forall \alpha \in \mathscr{A},\ \alpha \text{ is invariant},$$

then the proof will be completed in virtue of Theorem 1. Let $A_\xi \in \mathscr{A}$, where $\xi \in P$ is arbitrarily fixed. Let $y \in A_\xi$ and $s \in I_y$. The condition (c') in the definition of Lyapunov function ays down that $\lambda_s(y) \subset A_\xi$. The proof is completed.

In order to formulate the invers theorem, we shall have to make some additional assumptions for T. Namely, let us suppose that every set $\emptyset \neq Y \subset T$ has infimum and the assumption (H):

$$\text{(H)} \qquad \begin{cases} \forall \tau_1 \in P\ \exists \tau_2 \in P,\ \tau_2 < \tau_1, & \text{if } O \text{ occurs in } S \\ \forall \tau_1 \in P\ \exists \tau_2 \in P,\ \tau_2 > \tau_1, & \text{if } o \text{ occurs in } S \end{cases}$$

and

$$\forall \emptyset \neq Y \subset T\ \forall \xi \in P,\ (\inf Y \dot{<} \xi \Rightarrow \exists \bar{\xi} \in P:\ \bar{\xi} \leqslant \xi).$$

One of the examples of T is R_*, but there exist non-ordered spaces fulfilling the above assumption as well.

THEOREM 3. *If the family Ω is in the form, $\Omega = (\omega_\xi)_{\xi \in P}$, where $\omega_{\xi_1} \subset \omega_{\xi_2}$ for every $\xi_1 \leqslant \xi_2$, $\xi_1, \xi_2 \in P$, the set M is (S, Ω, β)-stable, then there exists a set $W \neq \emptyset$ and (S, Ω, β)-Lyapunov function V: $W \to T$ for the set M.*

Proof. We define

$$W = \{y \in X:\ R(y) \neq \emptyset\},$$
$$V(y) = \inf R(y), \qquad y \in W,$$

where $R(y) = \{\xi \in P:\ \lambda(y) \subset \omega_\xi\}$. Since the set M is (S, Ω, β)-stable, then $W \neq \emptyset$. We shall show that the family $\{A_\xi\}_{\xi \in P}$ fulfils (a). In virtue of the (S, Ω, β)-stability of the set M we have

$$S'\{\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (v \in I_\eta \underset{v}{\supset} \lambda(\lambda_v(\eta)) \subset \omega_\xi)]\},$$

where S' is the word obtained from the word S in such a way that in the place of the quantifier of the letter ω we put the corresponding quantifier of the letter ξ. Hence we have:

$$S'\{\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (v \in I_\eta \underset{v}{\supset} \lambda_v(\eta) \subset \{y \in X:\ \xi \in R(y)\})]\}.$$

Since $\{y \in X:\ \xi \in R(y)\} \subset A_\xi$, the condition (a) is satisfied. If we show the implication

$$\xi_1 < \xi_2 \Rightarrow A_{\xi_1} \subset \omega_{\xi_2},$$

$(A_\xi)_{\xi \in P}$ will fulfil condition (b) in virtue of assumptions (H).

Let $y \in A_{\xi_1}$, i.e. $\inf R(y) \leqslant \xi_1$. As $\inf R(y) < \xi_2$, thus there exists $\xi_3 \in R(y)$ such that $\xi_3 < \xi_2$. We have the following sequence of implications:

$$\xi_3 \in R(y) \Rightarrow \lambda(y) \subset \omega_{\xi_3} \Rightarrow y \in \omega_{\xi_3} \Rightarrow y \in \omega_{\xi_2}.$$

Thus condition (b) is satisfied. Now we shall show that (c') is satisfied. Let $y \in W$ (i.e. $R(y) \neq \emptyset$), $s \in I_y$, $z \in \lambda_s(y)$ be arbitrarily fixed. If we show that $R(z) \supset R(y)$, then

$z \in W$ and $V(z) \leqslant V(y)$. Let $\xi \in R(y)$ and $t \in I_z$. Then $t+s \in I_y$. Since $\xi \in R(y)$, thus $\lambda_{t+s}(y) \subset \omega_\xi$. In virtue of Definition 1 we have:

$$\lambda_t(t) \subset \lambda_{t+s}(y) ,$$

Thus $R(z) \supset R(y)$ and the proof of theorem is completed.

4. Now we shall give some applications of Theorems 2, 3 (and Remark 3). Namely, we shall construct the Lyapunov function for examples 1—3 in Section 2. We shall assume that $T = R_*$ in every example.

Example 1. The classical Lyapunov stability of a set M considered here is equivalent to the existence of a set $W \neq \emptyset$ and a function V: $W \to R_*$ such that:

$$(\bar{a}) \quad \forall x \in M \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall y \in K(x, \delta), \ y \in W \wedge V(y) \leqslant \varepsilon$$

$$(\bar{b}) \quad \forall \varepsilon > 0 \ \exists \delta > 0 \colon \{y \in W \wedge V(y) \leqslant \delta \Rightarrow \varrho(M, y) < \varepsilon\}$$

$$(c') \quad \forall y \in W \ \forall z \in \lambda(y), \ z \in W \wedge V(z) \leqslant V(y) .$$

Example 2. The set M is weakly stable if and only if there exists $W \neq \emptyset$ and V: $W \to R_*$ such that

$$(a) \quad \forall x \in M \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall y \in K(x, \delta) \ \exists s \in I_y \colon \{z \in \lambda_s(y) \Rightarrow z \in W \wedge V(z) \leqslant \varepsilon\}$$

$$(b) \quad \forall \varepsilon > 0 \ \exists \delta > 0 \colon \{y \in W \wedge V(z) \leqslant \delta \Rightarrow \varrho(M, y) < \varepsilon\}$$

$$(c') \quad \forall y \in W \ \forall z \in \lambda(y), \quad z \in W \wedge V(z) \leqslant V(y) .$$

Example 3. The set M is an attractor if and only if there exists $W \neq \emptyset$ and V: $W \to R_*$ such that

$$(\bar{a}) \quad \exists \delta > 0 \ \forall y \in K(M, \delta), \lim_{t \to \infty} V(\pi_t(y)) = 0$$

$$(\bar{b}) \quad \forall \varepsilon > 0 \ \exists \delta > 0 \colon \{y \in W \wedge V(y) \leqslant \delta \Rightarrow \varrho(M, y) < \varepsilon\}$$

$$(c') \quad \forall y \in W \ \forall z \in \pi(y), \quad z \in W \wedge V(z) \leqslant V(y) .$$

These examples 1—3 show that using Theorems 2 and 3 we can easily construct the corresponding Lyapunov function for every feasible word.

We shall give one more application of these theorems. First we shall recall the general definition of the attractor (given in paper [8]).

Let us suppose that (X, U, A, π) is a local pseudo-dynamical system. Let M be a non-empty subset of X and Ω, Λ non-empty families $\subset \mathscr{P}(X)$ such that $M \subset Q \cap L$ for every $Q \in \Omega$, $L \in \Lambda$.

Definition 5. We say that M is an (Ω, Λ)-*attractor* if and only if there exists $Q \in \Omega$ such that for every $x \in Q$, $L \in \Lambda$ there exists $s \in I_x$ such that $\pi(\pi_s(x)) \subset L$.

The following theorem gives a sufficient condition for M to be an attractor. This is the generalization of the classical theorem on the asymptotic stability of the compact sets. We shall call a family $\mathscr{B} \subset \mathscr{P}(E)$, $\mathscr{B} \neq \emptyset$ the *basis on* E if and only if for every $A, B \in \mathscr{B}$ there exists $C \in \mathscr{B}$ such that $C \subset A \cap B$ (this definition is taken from the university lectures of Professor S. Łojasiewicz). A point $y_0 \in E$ is said to be the limit point of basis $\mathscr{B}$ on E, if $y_0 \in \cap \{\bar{B}\colon B \in \mathscr{B}\}$.

In the proof of this theorem the following equivalent definition of a compact set in a topological space occurs:

$$E \text{ is compact} \Leftrightarrow \text{every basis on } E \text{ has a limit point.}$$

THEOREM 4. *Let us suppose that X is a topological space, the mappings $\pi_u\colon X \ni x \to \pi(u, x) \in X$ are continuous for every $u \in U$. Let $W \neq \emptyset$ be an invariant compact subset of the space X such that $W \supset Q$ for some $Q \in \Omega$. Let us assume that every $L \in \Lambda$ is open. If there exists a continuous function $\Phi\colon W \to R_*$ such that:*

(1) *$\Phi(x) = 0$ if and only if $x \in M$,*

(2) *$\Phi(\pi_t(x)) \leqslant \Phi(x)$ for every $x \in W$ and $t \in I_x$,*

(3) *for every $x \in W \setminus M$ there exists $s \in I_x$ such that $\Phi(\pi_s(x)) < \Phi(x)$,*

then M is a (Ω, Λ)-attractor.

Proof. Notice that M is a (Ω, Λ)-attractor if it is $(\text{ZdHOn}, \tilde{\Omega}, \tilde{\beta})$-stable, where $\tilde{\Omega} = \Lambda$ and $\tilde{\beta}(x) = \Omega$ for every $x \in X$. Therefore Theorem 2 is sufficient to show that the function $\Phi\colon W \to R_*$ fulfils:

(a) $\exists Q \in \Omega\ \forall x \in Q\ \forall \varepsilon > 0\ \exists_s \in I_x\colon\ \pi_s(x) \in W \wedge \Phi(\pi_s(x)) \leqslant \varepsilon$

(b) $\forall L \in \Lambda\ \exists \delta > 0\colon\ (x \in W,\ \Phi(x) \leqslant \delta \Rightarrow x \in L)$

(c′) $\forall y \in W\ \forall z \in \pi(y),\ z \in W \wedge \Phi(z) \leqslant \Phi(y)$.

We have (c′) immediately from (2). We shall now show (a). Under the assumptions there exists $Q \in \Omega$ such that $Q \subset W$. We shall put

$$\lambda(x) = \inf\{\Phi(\pi_s(x))\colon\ s \in I_x\}$$

for every $x \in Q$. Then the condition (a) can be formulated:

$$\forall x \in Q,\ \lambda(x) = 0\,.$$

Indeed, let us assume to the contrary that there exists $x_0 \in Q$ such that $\lambda = \lambda(x_0) > 0$. Thus $\{\Phi^{-1}([\lambda, \mu]) \cap \pi(x_0)\}_{\mu > \lambda}$ is a basis on W. Since W is compact, this basis has the limit point $y_0 \in W$, i.e. $y_0 \in \Phi^{-1}([\lambda, \mu]) \cap \overline{\pi(x_0)}$ for every $\mu > \lambda$. Hence we have $\Phi(y_0) = \lambda$, and using condition (1) we have $y_0 \notin M$. From condition (3) we obtain

$$\exists s \in I_{y_0}\colon\ \Phi(\pi_s(y_0)) < \Phi(y_0) = \lambda\,.$$

On the other hand

$$\pi(x_0) \subset \{z \in W\colon\ \Phi(z) \geqslant \lambda\}\,,$$

and because of the continuity of Φ

$$\overline{\pi(x_0)} \subset \{z \in W\colon\ \Phi(z) \geqslant \lambda\}\,.$$