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Limit sets in generalized pseudo-dynamical systems

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I. Introduction

The solutions of ordinary differential equations having the property of uniqueness are single valued mappings and generate dynamical systems in a known manner. Similarly single valued mappings are basic on the theory of pseudo-dynamical systems in the sense of [6]. The analysis of the solutions of differential equations without uniqueness and the analysis of the trajectories of orientor fields, however, lead to the consideration of multi valued mappings. In this instance a generalized dynamical system (E. A. Barbašin [1], Roxin [8]) is equivalent to a dynamical system. In the same sense the generalized pseudo-dynamical system introduced by A. Pelczar [7] is equivalent to a pseudo-dynamical system.

A triple (X, U, λ) is called a generalized pseudo-dynamical system in the sense [7], if X is an abstract space, $(U, +)$ is an abelian semi-group having the neutral element 0 and λ is such a mapping from $U\times X$ into $\mathscr{P}(X)$, the family of all the subsets of X, that $\forall u, v \in U$, $\forall x \in X$, $\forall A \in \mathscr{P}(X)$ the following conditions

$$\lambda_u(\lambda_v(A))\subset\lambda_{u+v}(A) = \lambda_{v+u}(A),\ \lambda_0(x) = \{x\}$$

are satisfied, when $\lambda_u(x) \overset{\text{df}}{=} \lambda(u, x)$ and $\lambda_u(A) \overset{\text{df}}{=} \bigcup_{x\in A} \lambda_u(x)$.

Generalized dynamical systems are particular examples of generalized pseudo-dynamical systems defined in this manner. It is also possible to consider pseudo-dynamical systems as examples of generalized pseudo-dynamical systems such that the single element sets and the elements of these sets are identified in these systems.

This paper deals with properties of some notions which are certain generalizations of the notion of limit set, which is known from the theory of dynamical systems. In the theory of generalized pseudo-dynamical systems it is possible to consider several variants of these generalizations. The variants presented here have some topological properties analogous with their prototype from the theory of dynamical systems. In the systems identified with some dynamical systems these variants are identical with "classical" limit sets. Moreover, the notion of a limit set of a set trajectory, presented in this paper, may be applied to characterize in a certain manner the periodicity of set trajectories defined in the generalized pseudo-dynamical systems.

II. Notations and basic definitions

Let (X, v) be a topological space with a topology v and let $(\mathscr{A}(X), 2^v)$ be the family of non-empty subsets of X, with the topology 2^v. The topology 2^v is generated by all families $\langle w_1, \dots, w_n \rangle^+ \overset{\text{df}}{=} \{E \in \mathscr{A}(X)\colon E \subset \bigcup_{i=1}^{n} w_i, E \cap w_i \neq \emptyset, \text{ for } i = 1, \dots, n\}$ corresponding to all the finite collections of open subsets of X in the topology v. [5].

Let σ be the mapping: $\mathscr{A}(\mathscr{A}(X)) \ni \mathscr{B} \to \bigcup_{E \in \mathscr{B}} E \in \mathscr{A}(X)$. By $\bar{\sigma}(\mathscr{B})$ denote the closure of set $\sigma(\mathscr{B})$ in the (X, v).

Let cl be the denotation of the closure operation (of a family of subsets of X in $(\mathscr{A}(x), 2^v)$.

Let (X, U, λ) denote a generalized pseudo-dynamical system in the sense of [7] and let $(X, (U, \prec), \lambda)$ denote a generalized pseudo-dynamical system with the added assumption that $(U, \prec)$ is a semi-group, directed by a relation "$\prec$" in agreement with the semi-group operation "$+$" and such that a neutral element 0 is the least element in $(U, \prec)$.

III. Limit sets definitions

Let a system $(X, (U, \prec), \lambda)$ be given. Assume that $A \in \mathscr{A}(X)$ and $x \in X$. In the beginning we make the following definitions:

Definition 1. The set $\lambda(A) \overset{\text{df}}{=} \bigcup_{u \in U} \lambda_u(A)$ is called *the emission range of set A*. The set $\lambda(x) \overset{\text{df}}{=} \lambda(\{x\})$ is called *the emission range of point x*.

Definition 2. The collection $T_\lambda(A) \overset{\text{df}}{=} \{\lambda_u(A), u \in U\}$ is called *the trajectory of set A*. The collection $T_\lambda(x) \overset{\text{df}}{=} T_\lambda(\{x\})$ is called *the trajectory of point x*.

Remark 1. Definition 1 implies that $\lambda(A) = \bigcup_{x \in A} \lambda(x)$ and Definition 2 implies that $T_\lambda(x) = \{\lambda_u(x)\colon u \in U\}$.

Remark 2. In Definitions 1 and 2 no advantage is taken of the directedness of $(U, \prec)$. Then it is possible in these definitions to replace the system $(X, (U, \prec), \lambda)$ by the system (X, U, λ). The sets and the collections of sets defined in this manner will be the same as in the system $(X, (U, \prec), \lambda)$.

Remark 3. In Definition 1 the expression "the emission range" is used while in [3] the analogous set in dynamical systems is called "the trajectory". In this paper the expression "the trajectory" will be used with reference to the collection of all sets $\lambda_u(A)$ or $\lambda_u(x)$, $u \in U$. Some collections $\{\lambda_u(A)\}_{u \in U}$ or $\{\lambda_u(x)\}_{u \in U}$ may be essentially different to some unions $\lambda(A)$ or $\lambda(x)$ of the sets belonging in these collections.

Example 1. Let us consider a family of all solutions $x(t)$ of equation $x' = 2\sqrt{x}$ for $t \geqslant 0$. Let a mapping $\lambda\colon R_0^2 \to \mathscr{A}(R_0^1)$, where $R_0^1 = [0, +\infty)$, be such that:

$$\forall t \geqslant 0 \ \forall x > 0\colon \lambda_t(0) \overset{\mathrm{df}}{=} [0, t^2]$$

and $\lambda_t(x) \overset{\mathrm{df}}{=} \{(t+\sqrt{x})^2\}$. In the system (R_0^1, R_0^1, λ), with a mapping defined in this manner, for every $x \in R_0^1$ the emission range of x is $\lambda(x) = [x, +\infty)$. For every $x > 0$ this is the union of the one-element sets $\{(t+\sqrt{x})^2\}$, where $t \in R_0^1$ and it may be identified with the family of these sets, as is often done by an identification of every one-element set with its element, but then the emission range of $x = 0$ is an union of the intervals $[0, t^2]$, where $t \in R_0^1$ and it is essentially dissimilar to the trajectory of $x = 0$.

Example 2. Let us consider a system (R^1, R_0^1, λ), where $R^1 = (-\infty, +\infty)$ $R_0^1 = [0, +\infty)$ and λ is a mapping such that for every integer z and for each $t \in R_0^1$ if $q \in [z, z-1)$, then $\lambda_0(q) \overset{\mathrm{df}}{=} \{q\}$ and $\lambda_t(q) \overset{\mathrm{df}}{=} [q, z+1)$ for $t > 0$. In this case for each $q \in [z, z+1)$, where z is an integer, the trajectory of q is a family composed of two sets: $\{q\}$ and $[q, z+1)$ while the emission range of q is one set $[q, z+1)$.

Example 3. Let us consider the family of all solutions $(x(t), y(t))$ of a system of equations $x' = -x$, $y' = y$. Let $\lambda\colon R_0^2 \to \mathscr{A}(R_0^2)$ be a mapping defined as follows:

$$\forall t \geqslant 0 \ \forall (x, y) \in R_0^2\colon \lambda_t((x, y)) \overset{\mathrm{df}}{=} [xe^{-t}, x] \times [y, ye^t]\,.$$

For every point $(x, y) \in R_0^2$ in the system $(R_0^1, (R_0^1, \leqslant), \lambda)$ with the mapping λ defined in this manner, the set $(0, x] \times [y, +\infty)$ is the emission range of this point and the collection $\{[xe^{-t}, x] \times [y, ye^t]\}_{t \in R_0^1}$ is its trajectory. In particular the trajectory and the emission range of the point $\mathcal{O}(0, 0)$ are the same one-element set $\{(0, 0)\}$.

Assume that a system $(X, (U, \prec), \lambda)$ is given and assume that $A \in \mathscr{A}(X)$ and $x \in X$. Let us put the following:

Definition 3. The collection $\Omega_*(A) \overset{\mathrm{df}}{=} \bigcup\limits_{u \in U} (cl\lambda_v(A)\colon u \prec v, v \in U\})$ is called *$\prec$-limit set of the trajectory of set A*. The collection $\Omega_*(x) = \Omega_*(\{x\})$ is called *$\prec$-limit set of the trajectory of point x*.

Definition 4. The set $L_*(A) \overset{\mathrm{df}}{=} \sigma(\Omega_*(A))\ (= \bigcup\limits_{E \in \Omega_*(A)} E)$ is called *$\prec$-limit set of set A*. The set $L_*(x) = L_*(\{x\})$ is called *$\prec$-limit set of point x*.

Definition 5. The union of $\prec$-limit sets of all points of the set $A \in \mathscr{A}(X)$ is called *$\prec$-limit set of all points of this set A* and is denoted by $\Lambda_*(A)$.

Remark 4. From the Definitions 3, 4 and 5 it follows that

$$\Omega_*(x) = \bigcap_{u \in U} (cl\{\lambda_v(x)\colon u \prec v, v \in U\}),$$

$$L_*(x) = \sigma(\Omega_*(x)) \ \ (= \bigcup_{E \in \Omega_*(\{x\})} E),$$

$$\Lambda_*(A) = \bigcup_{x \in A} L_*(x),$$

$$\Lambda_*(x) = L_*(x), \text{ where } \Lambda_*(x) \text{ denotes } \Lambda_*(\{x\})\,.$$

Definition 6. The set $G_*(A) \overset{\mathrm{df}}{=} \bigcup_{u\in U} \bar{\sigma}(\{\lambda_v(A)\colon u\prec v,\, v\in U\})$ is called $\prec$*-limit set of emission range of set A*. The set $G_*(x) = G_*(\{x\})$ is called $\prec$*-limit set of emission range of point x*.

Definition 7. The union of $\prec$-limit sets of all points of some set $A\in\mathscr{A}(X)$ is called $\prec$*-limit set of all emission ranges of points of this set A* and is denoted by $\Gamma_*(A)$.

Remark 5. From the Definitions 6 and 7 it follows that

$$G_*(x) = \bigcap_{u\in U} \bar{\sigma}(\{\lambda_v(x)\colon u\prec v,\, v\in U\}),$$

$$\Gamma_*(A) = \bigcup_{u\in A} G_*(x),$$

$$\Gamma_*(x) = G_*(x), \text{ where } \Gamma_*(x) \text{ denotes } \Gamma_*(\{x\}).$$

Example 4. Let us consider for every $t_0\geqslant 0$, $x_0\geqslant 0$ all such solutions of equation $x'(t) = 2\sqrt{x(t)}$ for which the initial condition $x(t_0) = x_0$ is satisfied. For these solutions we may define a system $(R_0^2, (R_0^1, \leqslant), \lambda)$ such that:

$$\forall t\geqslant 0\ \forall t_0\geqslant 0\ \forall x_0>0\colon \lambda_t((t_0, x_0)) \overset{\mathrm{df}}{=} \{(p, q)\}, \text{ where } p = t_0+t, \quad q = (t+x_0)^2,$$

$$\forall t\geqslant 0\ \forall t_0\geqslant 0\colon \lambda_t((t_0, 0)) \overset{\mathrm{df}}{=} \{(p, q)\in R_0^2\colon p = t_0+t,\, 0\leqslant q\leqslant t^2\}$$

and the relation "$\leqslant$" is the weak inequality between the real numbers. For each set $A\in\mathscr{A}(R_0^2)$ all its limit sets are empty in this system.

Example 5. If in the system (R_0^1, R_0^1, λ), defined in Remark 2, we shall put the directed semi-group $(R_0^1, \leqslant)$ in place of the semi-group R_0^1 then for every $A\in\mathscr{A}(R_0^1)$, such that $0\in A$, the collection $\Omega_*(A)$ will be composed of one set R_0^1. Then $\Lambda_*(A) = L_*(A) = R_0^1$. Also obviously $\Gamma_*(A) = G_*(A) = R_0^1$. If $0\notin A$, then all limit sets are empty.

Example 6. In the system $(R^2, (R_0^1, \leqslant), \lambda)$ let the mapping λ be such that $\forall t\geqslant 0\ \forall (x, y)\in R^2\colon \lambda_t((x, y)) \overset{\mathrm{df}}{=} \{(p, q)\in R^2\colon p = xe^{-t},\ |q-y|\leqslant t\}$. Then for every set $A\in\mathscr{A}(X)$ the collection $\Omega_*(A)$ is composed of one set $\{0\}\times R^1$ and the other limit sets are all equal to this set.

IV. Some relations between the presented variants of limit sets

1. For every set $A\in\mathscr{A}(X)$ the collection $\Omega_*(A)$ is obviously different from all sets $L_*(A)$, $\Omega_*(A)$, $\Gamma_*(A)$ and $G_*(A)$, as a family of some subsets of X.

Example 7. The formulae

$$(*)\qquad \begin{aligned} x(t) &= x_0\cos t - y_0\sin t,\\ y(t) &= x_0\sin t + y_0\cos t \end{aligned}$$

define solutions of the system of differential equations

$$x'(t) = -y(t)\,,$$
$$y'(t) = x(t)$$

such that there are fulfilled the initial conditions $x(0) = x_0$, $y(0) = y_0$. Let us consider a system $(R^2, (R_0^1, \leqslant), \lambda)$ such that for every $(x_0, y_0) \in R^2$ and for every $t \in R_0^1$ is $\lambda_t((x_0, y_0)) \overset{\mathrm{df}}{=} \{(x(t), y(t))\}$, where $x(t)$, $y(t)$ are calculated by the formulae (*). Let us put $A = \{0\} \times [0, 1]$. In this system the collection $\Omega_*(A)$ is composed of sets:

$$\lambda_t(A) = \{(x(t), y(t)) \in R^2\colon\ x(t) = -y_0 \sin t, y(t) = y_0 \cos t, y_0 \in [0, 1], t \in R_0^1\}\,.$$

The other limit sets for A are equal to the set $\{(x, y) \in R^2\colon\ x^2+y^2 \leqslant 1\}$.

Example 8. The trajectories of the differential equations system $x'(t) = -x(t)-y(t)$, $y'(t) = x(t)-y(t)$, such that they run across some points $(x_0, y_0) \in R^2$ different from $(0, 0)$, are spirals convergent towards the point $(0, 0)$ where $t \to +\infty$. Let $(x_0, y_0) \neq (0, 0)$ be given. By $(x_0(t), y_0(t))$ denote the image of point (x_0, y_0) after a time t on the spiral running across (x_0, y_0). Let a system $(R^2, (R_0^1, \leqslant), \lambda)$ be such that for every $t \in R_0^1$ and for every point $(x_0, y_0) \in R^2$, different from $(0, 0)$, a set $\lambda_t((x_0, y_0))$ is a closed segment of the line running across $(0, 0)$ and $(x_0(t), y_0(t))$ such that it is included between $(x_0(t), y_0(t))$ and a point of the circumference $x^2+y^2 = x_0^2+y_0^2$. Let us put $\lambda_t((0, 0)) \overset{\mathrm{df}}{=} \{(0, 0)\}$ for every $t \in R_0^1$. In this system for every $(x_0, y_0) \neq (0, 0)$ the collection $\Omega_*((x_0, y_0))$ is a family of radius vectors of points on the circumference $x^2+y^2 = x_0^2+y_0^2$. The other limit sets are equal to the circle $x^2+y^2 \leqslant x_0^2+y_0^2$.

2. The relation $G_*(A) \supset L_*(A)$ is always fulfilled. Indeed, it follows from the definitions that

$$G_*(A) = \bigcap_{u \in U} \bar{\sigma}\,\{\bigcup_{x \in A} \lambda_v(x)\colon\ u \prec v, v \in U\} \text{ and}$$

$$L_*(A) = \sigma \bigcap_{u \in U} cl\{\bigcup_{x \in A} \lambda_v(x)\colon\ u \prec v, v \in U\}\,.$$

Hence we have only to observe that for every $u_0 \in U$ the following inclusions are fulfilled:

$$\sigma \bigcap_{u \in U} cl\{\bigcup_{x \in A} \lambda_v(x)\colon\ u_0 \prec v, v \in U\} \subset \bigcap_{u \in U} \sigma cl\{\bigcup_{x \in A} \lambda_v(x)\colon\ u_0 \prec v, v \in U\} \text{ and}$$

$$\bar{\sigma}\{\bigcup_{x \in A} \lambda_v(x)\colon\ u_0 \prec v, v \in U\} \supset \sigma cl\{\bigcup_{x \in A} \lambda_v(x)\colon\ u_0 \prec v, v \in U\}\,.$$

These inclusions and the two preceding identities imply the inclusion $G_*(A) \supset L_*(A)$.

3. For the sets $\Lambda_*(A)$, $\Gamma_*(A)$ and $G_*(A)$ the following inclusions are always fulfilled: $\Lambda_*(A) \subset \Gamma_*(A) \subset G_* A$. Indeed, the definitions of these sets imply the following identities and inclusions:

$$\Lambda_*(A) = \bigcup_{x \in A} L_*(x), \ \Gamma_*(A) = \bigcup_{x \in A} G_*(x),$$

$$\Gamma_*(A) \subset \bigcap_{u \in U} \bigcup_{x \in A} \bar{\sigma}\{\lambda_v(x) \colon u \prec v, v \in U\},$$

$$G_*(A) = \bigcap_{u \in U} \bar{\sigma}\{\bigcup_{x \in A} \lambda_v(x) \colon u \prec v, v \in U\},$$

$$\bigcup_{x \in A} \bar{\sigma}\{\lambda_v(x) \colon u_0 \prec v, v \in U\} \subset \bar{\sigma}\{\bigcup_{x \in A} \lambda_v(x) \colon u_0 \prec v, v \in U\} \text{ for each } u_0 \in U.$$

4. Some sets $\Lambda_*(A)$, $L_*(A)$, $\Gamma_*(A)$ and $G_*(A)$ may be non-identical. This is illustrated by the following examples.

Example 9. Let (X, v) be a T_1-topological space. Let $(U, \prec)$ be $(R_0^1, \leqslant)$ and let $\lambda_u(x) \overset{\text{df}}{=} \{x\}$ for every $u \in U$ and for every $x \in X$ in the system $(X, (U, \prec), \lambda)$. Then for each open set $A \in \mathscr{A}(X)$ we have $\Lambda_*(A) = \Gamma_*(A) = A$ and $G_*(A) = L_*(A) = \bar{A}$.

Example 10. Let us put $(R_0^1, \leqslant)$ in the place of R_0^1 in the system (R_0^2, R_0^1, λ) defined in Example 3. Then for every point $z = (x, y) \in R_0^2$ the collection $\Omega_*(z)$ is composed of one set $[0, x] \times [y, +\infty)$. This set is equal to every set of the other limit sets. For the set $A = \{(p, q) \in R_0^2 \colon p = x, y_1 < q < y_2, x, y_1, y_2 \in R_0^1\}$ the collection $\Omega_*(A)$ has the union equal to the set $[0, x] \times [y_1, +\infty)$ and this set is equal to $L_*(A)$ and $G_*(A)$. Simultaneously $\Lambda_*(A) = \Gamma_*(A) = [0, x] \times (y_1, +\infty)$.

Example 11. Let (X, v) and $(U, \prec)$ be as in Example 9 and let the mapping λ in the system $(X, (U, \prec), \lambda)$ be such that for every x of an open set $A \in \mathscr{A}(X)$ and for an $u_0 \neq 0$ the inclusion $\lambda_u(x) \subset A$ is fulfilled for $u \prec u_0$ and the identity $\lambda_u(x) = A$ is fulfilled for $u_0 \prec u$. Then $L_*(A) = \Lambda_*(A) = \Gamma_*(A) = G_*(A) = \bar{A}$.

Example 12. Let (X, v) be a space R^1 and let $(U, \prec)$ be $(R_0^1, \leqslant)$. The system $(X, (U, \prec), \lambda)$ we define as follows: $\lambda_u(x) \overset{\text{df}}{=} \{x\}$ for each $x \neq 0$ and for each $u \geqslant 0$, but if $x_0 = 0$, then $\lambda_0(x_0) = \{0\}$ and $\lambda_u(x_0) = [0, 1 - e^{-u})$ for $u > 0$. For the set $A = (-1, 1)$ in this system we have $\Lambda_*(A) = \Gamma_*(A) = (-1, 1]$ and $L_*(A) = G_*(A) = [-1, 1]$. Hence $\Lambda_*(A) = \Gamma_*(A) \neq G_*(A) = L_*(A)$. Simultaneously for $A_1 = [-1, 1)$ we have $\Lambda_*(A_1) = \Gamma_*(A_1) = G_*(A_1) = L_*(A_1)$.

V. Some topological properties of limit sets

1. Invariantness. Let us introduce the following definitions:

Definition 8. A collection $\mathscr{B} \subset \mathscr{A}(X)$ is called *invariant in* $(X, (U, \prec), \lambda)$ if and only if $\forall u \in U \ \forall B \in \mathscr{B} \colon \lambda_u(B) \in \mathscr{B}$.

Definition 9. A set $Z \subset X$ is called *invariant in* $(X, (U, \prec), \lambda)$ if and only if $\forall u \in U \; \forall z \in Z$: $\lambda_u(z) \subset Z$.

Remark 6. The trajectory of every set is an invariant collection of sets and the emission range of every set is an invariant set in every generalized pseudo-dynamical system.

Let a system $(X, (U, \prec), \lambda)$ be given and let $A \in \mathscr{A}(X)$. The following two theorems give sufficient conditions for the invariantness of the collection $\Omega_*(A)$ and the sets $\Lambda_*(A)$, $L_*(A)$, $\Gamma_*(A)$ and $G_*(A)$.

THEOREM 1. *Let* $(\mathscr{A}(X), 2^v)$ *be a Hausdorff space. Suppose that a system* $(X, (U, \prec), \lambda)$ *has the following properties:* $\forall u \in U$ *the mapping* $\lambda_u(\cdot)$: $\mathscr{A}(X) \ni B \mapsto \lambda_u(B) \in \mathscr{A}(X)$ *is continuous on* $\mathscr{A}(X)$ *and* $\forall A \in \mathscr{A}(X) \; \forall u, v \in U$:

$$\lambda_u(\lambda_v(A)) = \lambda_{u+v}(A) = \lambda_v(\lambda_u(A)) .$$

Then for $A \in \mathscr{A}(X)$ *the collection* $\Omega_*(A)$ *is invariant and the sets* $\Lambda_*(A)$ *and* $L_*(A)$ *are invariant too.*

Proof: Let $B \in \Omega_*(A)$. Since $(U, \prec)$ is directed, there exists an M-S sequence (Moore-Smith's sequence) $\{\lambda_v(A)\}$ convergent to B such that v are some elements of a subset of U, being co-final with $(U, \prec)$. For every $p \in U$ the M-S sequence $\{\lambda_t(A)\}$, where $t = p+v$, $p = \text{const.}$, is convergent to $\lambda_p(B)$; thus $\lambda_p(B) \in \Omega_*(A)$. Hence $\Omega_*(A)$ is invariant.

Let $z \in L_*(A)$. Then there exists a set $B \in \Omega_*(A)$ such that $z \in B$. By the definition of $\lambda(z)$ and $\lambda(B)$ the inclusion $\lambda(z) \subset \lambda(B)$ holds true. By the invariantness of $\Omega_*(A)$ we obtain $\lambda(B) \subset L_*(A)$ and then $\lambda(z) \subset L_*(A)$. Hence $L_*(A)$ is invariant. The set $\Lambda_*(A)$ is invariant as a union of invariant sets $L_*(\{x\})$, $x \in A$.

THEOREM 2. *Let* $(\mathscr{A}(X), 2^v)$ *be a Hausdorff space. Assume that in a system* $(X, (U, \prec), \lambda)$ *the mapping* $\lambda_v(\cdot)$: $X \in x \to \lambda_v(x) \in \mathscr{A}(X)$ *is continuous on* X *for every* $v \in U$. *Then for every* $A \in \mathscr{A}(X)$ *the sets* $G_*(A)$ *and* $\Gamma_*(A)$ *are invariant.*

Proof: Let $z \in G_*(A)$. There exists an M-S sequence $\{x_u\}$ convergent to z such that u are elements of a subset of U, being co-final with $(U, \prec)$ and respectively $x_u \in \lambda_u(A)$. Let $p \in U$ be given. By the continuity of $\lambda_p(-)$ we obtain that the M-S sequence $\{\lambda_p(x_u)\}$ is convergent to $\lambda_p(z)$. Since $\lambda_p(x_u) \subset \lambda_{p+u}(A)$ every point of $\lambda_p(z)$ belongs in $G_*(A)$. Hence $\lambda_p(z)$ is included in $G_*(A)$ and then $G_*(A)$ is invariant for every $A \in \mathscr{A}(X)$. In particular for one-element sets $\{a\}$, the sets $G_*(\{a\})$ are invariant also. Hence the set $\Gamma_*(A)$ is invariant as a union of invariant sets.

Remark 7. Let the equation $x'(t) = f(t, x(t))$ with the initial condition $x(0) = x_0$ have some unique solutions defined for every $x_0 \in R^n$ on the whole of R^1. Let a mapping λ such that for every $x_0 \in R^n$ and for every $t \in R^1_*$ is $\lambda_t(x_0) \overset{\mathrm{df}}{=} \{y \in R^n$: $y = x(t)$, or $y = x(\tau)$, $|\tau| \leqslant t$, or $y = x(\tau)$, $0 \leqslant \tau \leqslant t$, where $x(\cdot)$ is a solution of $x'(t) = f(t, x(t))$ such that $x(0) = x_0\}$, defines a generalized pseudo-dynamical system in R^n. In such a system the assumption from Theorem 2 on the continuity of some mappings $\lambda_v(-)$ is fulfilled. Moreover this assumption is fulfilled in the whole of systems $(X, (U, \prec), \lambda)$ generated by an identification of some points of dynamical systems with some one-element sets in adequate generalized pseudo-dynamical systems.

If the assumptions of Theorems 1 and 2 on the continuity of mappings $\lambda_u(\cdot)$ and $\lambda_u(-)$ for $u \in U$ are not fulfilled in a system $(X, (U, \prec), \lambda)$, then for an $A \in \mathscr{A}(X)$ the sets $\Lambda_*(A)$, $L_*(A)$, $\Gamma_*(A)$, $G_*(A)$ and the family $\Omega_*(A)$ may be non-invariant.