

## Remarks on derivations on normed algebras

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I. M. Singer and J. Wermer have shown in [1] that if  $d$  is a continuous derivation on a commutative Banach algebra  $A$  with unit then the set of values of  $d$  is contained in the radical of  $A$ . The proof is based upon some properties of special entire function with values in the class of all automorphisms of  $A$ . E. A. Gorin, J. J. Kotchetkov and B. S. Mitiagin have proved in [2] a more general result using an entirely different simple algebraic method.

In this paper we shall study similar problems but for a considerably larger family of mappings.

Notation: By  $N, R$  and  $C$  we mean the natural, real and complex numbers respectively. Let  $A$  be a ring,  $x, y \in A$  and let  $f, g, h$  be mappings of  $A$  into  $A$ . Then we define:  $[x, y] = xy - yx$ ,  $[f, g] = f \circ g - g \circ f$ ,  $h^n = h \circ h \circ \dots \circ h$  ( $n$ -times) and  $h^0(x) = x$ . We also use the notation  $\text{rad}(A)$  for the radical of  $A$ .

**Definition 1.** Let  $A$  be a ring (or an algebra over some field  $K$ ). The mapping  $d: A \rightarrow A$  is called a *derivation on type*  $(f, g)$  if:

- (i)  $f, g$  are endomorphisms of  $A$ ,
- (ii)  $d(xy) = f(x)d(y) + d(x)g(y)$ ,
- (iii)  $d(x+y) = d(x) + d(y)$  (and  $d(\lambda x) = \lambda d(x)$ ),

for each  $x, y \in A$  (and  $\lambda \in K$ ). If  $f(x) = g(x) = x$  for all  $x \in A$  then  $d$  is called a *derivation*.

**Definition 2.** Let  $E$  be a non-empty subset of a some ring  $A$ . Then the set:

$$\hat{E} = \{x \in A: \exists n \in N \exists y \in E \text{ such that } x = y^n\}$$

is called the *power envelope* of  $E$ .

We shall first prove some purely algebraic lemmas.

**LEMMA 1.** Let  $d$  be a derivation on type  $(f, g)$  on a ring  $A$ . Let  $V_n^m$  denote the set of all mappings of  $\{1, 2, \dots, m\}$  into  $\{1, 2, \dots, n\}$ . We define  $q_j^n = \underbrace{f \times \dots \times f}_{(j-1)\text{-times}} \times d \times \underbrace{g \times \dots \times g}_{(n-j)\text{-times}}$ ,  $P_n: A^n \ni (a_1, \dots, a_n) \mapsto a_1 a_2 \dots a_n \in A$ ,  $Q_v^n = P_n \circ q_{v(1)}^n \circ \dots \circ q_{v(m)}^n$  where  $v \in V_n^m$ . Then:

$$(1) \quad d^m(x_1 x_2 \dots x_n) = \sum_{v \in V_n^m} Q_v^n(x_1, x_2, \dots, x_n)$$

where  $x_1, x_2, \dots, x_n \in A$ .

Proof. For every fixed index  $n$  we have the following formula (we omit the trivial proof):

$$(2) \quad d(x_1 x_2 \dots x_n) = \sum_{j=1}^n f(x_1) \dots f(x_{j-1}) d(x_j) g(x_{j+1}) \dots g(x_n)$$

The general case follows by induction on  $m$ . Suppose that the formula (1) is true for some  $m$ . Then:

$$\begin{aligned} d^{m+1}(x_1 x_2 \dots x_n) &= \sum_{v \in V_n^m} d \circ P_n \circ q_{v(1)}^n \circ \dots \circ q_{v(m)}^n(x_1, \dots, x_n) \\ &= \sum_{v \in V_n^m} \sum_{j=1}^n P_n \circ q_j^n \circ q_{v(1)}^n \circ \dots \circ q_{v(m)}^n(x_1, \dots, x_n) = \sum_{v \in V_n^{m+1}} Q_v^n(x_1, \dots, x_n) \end{aligned}$$

LEMMA 2. Let  $A$  be a ring with unit and  $d$  be a derivation on type  $(f, g)$  on  $A$ . If  $x$  is a given point in  $A$ , we define:

$$(3) \quad G_n^k(x) = \begin{cases} \sum_{m=0}^{k-1} (-1)^m \binom{k}{m} f^n(x^m) d^n(x^{k-m}), & \text{if } f = g \\ \sum_{m=0}^{k-1} \sum_{r=0}^m (-1)^m \frac{\binom{k}{m}}{m+1} f^n(x^r) d^n(x^{k-m}) g^n(x^{m-r}), & \text{if } f \neq g. \end{cases}$$

Suppose that  $[f, d] = [g, d] = 0$ ,  $[f(x), d(x)] = [g(x), d(x)] = 0$  and that

$$f | f(A)d(A)g(A) = g | f(A)d(A)g(A).$$

Then we have the following properties:

- (4) if  $f = g$  or  $[d(f^m(x)), d^m(f(x^s))] = 0$  (for all  $s, m \in N$ ) then  $d(G_n^k(x)) = G_{n+1}^k(x) - kd(f^n(x))f(G_n^{k-1}(x))$
- (5) if  $f = g$  and  $k > n$  then  $G_n^k(x) = 0$
- (6) if  $f = g$  or  $[d(f^m(x)), d^m(f(x^s))] = d(G_n^{m+1}(x)) = 0$  (for all  $s, m \in N - \{0\}$ ) then  $G_n^n(x) = n! f^{n-1}(d(x))^n$ .

Proof. If we take  $f = g$  then we obtain:

$$\begin{aligned} d(G_n^k(x)) &= \sum_{m=0}^{k-1} (-1)^m \binom{k}{m} (f^{n+1}(x^m) d^{n+1}(x^{k-m}) + d(f^n(x^m)) f(d^n(x^{k-m}))) \\ &= G_{n+1}^k(x) + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} m f^{n+1}(x^{m-1}) d(f^n(x)) f(d^n(x^{k-m})) \\ &= G_{n+1}^k(x) - k \sum_{m=0}^{k-2} (-1)^m \binom{k-1}{m} d(f^n(x)) f(f^n(x^m) d^n(x^{k-m-1})) \\ &= G_{n+1}^k(x) - kd(f^n(x)) f(G_n^{k-1}(x)). \end{aligned}$$

Assume now that  $f \neq g$ . We have:

$$\begin{aligned}
 d(G_n^k(x)) &= G_{n+1}^k(x) + \sum_{m=1}^{k-1} \sum_{r=1}^m (-1)^m \frac{\binom{k}{m}}{m+1} d(f^n(x^r)) g(d^n(x^{k-m})) g^{n+1}(x^{m-r}) + \\
 &+ \sum_{p=1}^{k-1} \sum_{q=0}^{p-1} (-1)^p \frac{\binom{k}{p}}{p+1} f^{n+1}(x^q) f(d^n(x^{k-p})) d(g^n(x^{p-q})) \\
 &= G_{n+1}^k(x) + \sum_{m=1}^{k-1} \sum_{r=1}^m (-1)^m \frac{\binom{k}{m}}{m+1} \times \\
 &\quad \times \left( \sum_{s=0}^{r-1} d(f^n(x)) f^{n+1}(x^s) g(f^n(x^{r-s-1})) g(d^n(x^{k-m})) g^n(x^{m-r}) \right) \\
 &+ \sum_{p=1}^{k-1} \sum_{q=0}^{p-1} (-1)^p \frac{\binom{k}{p}}{p+1} f(f^n(x^q) d^n(x^{k-p})) \left( \sum_{i=0}^{p-q-1} d(g^n(x)) f(g^n(x^i)) g^{n+1}(x^{p-q-i-1}) \right) \\
 &= G_{n+1}^k(x) + \sum_{m=1}^{k-1} \sum_{r=1}^m (-1)^m \frac{\binom{k}{m}}{m+1} \frac{r}{m+1} d(f^n(x)) g(f^n(x^{r-1})) d^n(x^{k-m}) g^n(x^{m-r}) + \\
 &+ \sum_{p=1}^{k-1} \sum_{q=0}^{p-1} (-1)^p \frac{\binom{k}{p}}{p+1} \frac{p-q}{p+1} d(g^n(x)) f(f^n(x^q) d^n(x^{k-p})) g^n(x^{p-q-1}) \\
 &= G_{n+1}^k(x) + \sum_{m=1}^{k-1} \sum_{r=0}^{m-1} (-1)^m \frac{\binom{k}{m}}{m} d(f^n(x)) f(f^n(x^r) d^n(x^{k-m})) g^n(x^{m-r-1}) \\
 &= G_{n+1}^k(x) - kd(f^n(x)) f(G_n^{k-1}(x)).
 \end{aligned}$$

The proof of (4) has been completed. Next we prove (5). We first take  $n = 1$ . When  $k > 1$  we have:

$$\begin{aligned}
 G_1^k(x) &= \sum_{m=0}^{k-1} (-1)^m \binom{k}{m} f(x^m) (k-m) f(x^{k-m-1}) d(x) = kf(x^{k-1}) d(x) \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \\
 &= kf(x^{k-1}) d(x) (1-1)^{k-1} = 0.
 \end{aligned}$$

The general case follows by induction on  $n$  and by formula (4). For the case  $n = 1$  we have  $G_1^1(x) = d(x)$ . Suppose that (6) is true for some  $n$ . Applying (4), (5) and the assumptions of (6) we obtain:

$$\begin{aligned}
 G_{n+1}^{n+1}(x) &= +d(G_n^{n+1}(x)) + (n+1)d(f^n(x)) f(G_n^n(x)) = (n+1)! f^n(d(x)) f^n(d(x))^n \\
 &= (n+1)! f^n(d(x))^{n+1}.
 \end{aligned}$$

The proof of lemma 2 is complete.

In the case when for all  $x \in A$  we have  $f(x) = g(x) = x$ , the result given in Lemma 2 reduces to the result of E. A. Gorin, J. J. Kotchetkov and B. S. Mitiagin (see [2]). Now we can state and prove our main result.

**THEOREM 1.** *Let  $A$  be a normed algebra with unit over the real or complex field and let  $x \in A$ . Suppose that  $d$  is a continuous derivation on type  $(f, g)$  on  $A$  with the following properties:*

- (7)  $[f, d] = [g, d] = 0$ ,
- (8)  $f, g$  are continuous and  $f$  is bounded from below on the power envelope of the set  $d(A)$ ,
- (9)  $[f(x), d(x)] = [g(x), d(x)] = 0$ ,
- (10)  $f(y) = g(y)$  whenever  $y \in f(A)d(A)g(A)$ ,
- (11) if  $f \neq g$  then  $d$  satisfies supplementarily  $[d^m(f(x)), f^m(d(x^s))] = d(G_m^{m+1}(x)) = 0$ .

Then

$$(12) \quad \lim_{n \rightarrow \infty} \|(d(x))^n\|^{1/n} = 0.$$

*Proof.* The mapping  $f$  is bounded from below on the set  $\widehat{d(A)}$  i.e. there is a positive constant  $M$  such that  $M\|y\| \leq \|f(y)\|$  for every  $y \in \widehat{d(A)}$ . Then the following inequalities hold true:

$$\begin{aligned} n!(\min\{1, M\})^n \|(d(x))^n\| &\leq n!(\min\{1, M\})^{n-1} \|(d(x))^n\| \\ &\leq n!M^{n-1} \|(d(x))^n\| \leq n!M^{n-2} \|(f(d(x)))^n\| \leq \dots \\ &\leq n!M \|f(d(f^{n-3}(x)))\| \leq n! \|f(d(f^{n-2}(x)))\| = \|G_n^n(x)\|. \end{aligned}$$

Assume first that  $f \neq g$ . In this case we have:

$$\begin{aligned} \|G_n^n(x)\| &\leq \sum_{m=0}^{n-1} \sum_{r=0}^m \frac{\binom{n}{m}}{m+1} \|f^n(x^r) d^m(x^{n-m}) g^n(x^{m-r})\| \\ &\leq \sum_{m=0}^{n-1} \sum_{r=0}^m \frac{\binom{n}{m}}{m+1} \|f\|^n \|d\|^m \|g\|^n \|x\|^n = \|f\|^n \|d\|^n \|g\|^n \|x\|^n \left( \sum_{m=0}^{n-1} \binom{n}{m} \right) \\ &= \|f\|^n \|d\|^n \|g\|^n \|x\|^n (2^n - 1) \leq (2\|f\| \|d\| \|g\| \|x\|)^n. \end{aligned}$$

If we take  $f = g$  then we obtain:

$$\|G_n^n(x)\| \leq \sum_{m=0}^{n-1} \binom{n}{m} \|f^n(x^m) d^m(x^{n-m})\| \leq (2\|f\| \|d\| \|x\|)^n.$$

Therefore we have the inequality:

$$\|(d(x))^n\|^{1/n} \leq 2 \frac{\max\{1, \|g\|\} \|f\| \|d\| \|x\|}{\min\{1, M\}} \left(\frac{1}{n!}\right)^{1/n}.$$

The limit passage in the last inequality completes the proof.

The following very simple example is an illustration of this situation.

Example 1. Let  $A = R^2 = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x, y \in R \right\}$  with standard addition and multiplication of matrices. Let  $a, b \in R - \{0\}$  such that  $a^2 \neq b^2$  and let

$$\mathcal{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \mathcal{A}^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}$$

We define:

$$\begin{aligned} f: A \ni \mathcal{B} &\mapsto \mathcal{A}\mathcal{B}\mathcal{A}^{-1} \in A, \\ g: A \ni \mathcal{B} &\mapsto \mathcal{A}^{-1}\mathcal{B}\mathcal{A} \in A, \\ d &= f - g, \end{aligned}$$

$d$  has the properties (7), (8), (9), (11) and  $d(\text{rad}(A)) = \text{rad}(A)$ .

Note that the assumption (7) in Theorem 1 is substantial.

Example 2. Let  $D$  be the open disc in the complex plane with centre at  $a \in \mathbb{C}$  and radius  $r > 0$ . The collection of all holomorphic functions in  $D$  will be denoted by  $H(D)$ .  $H(D)$  is the algebra with pointwise addition and multiplication. It is normed with:

$$\|f\| = \sup \left\{ |f(z)| : |z - a| \leq \frac{r}{2} \right\}.$$

Define

$$\begin{aligned} J: \mathbb{C} \ni z_0 &\mapsto (D \ni z \mapsto z_0 \in \mathbb{C}) \in H(D) \\ \tilde{h}: H(D) \ni f &\mapsto f(a) \in \mathbb{C} \\ \tilde{d}: H(D) \ni f &\mapsto f'(a) \in \mathbb{C} \\ d &= J \circ \tilde{d}, \quad h = J \circ \tilde{h} \end{aligned}$$

Of course  $d$  is  $(h, h)$ -derivation on  $H(D)$ . It is continuous by Cauchy's Estimates. The properties (8), (9) hold true for every  $f \in H(D)$ . Let  $f_0 \in H(D)$  and  $f'_0(a) \neq 0$ . Then  $[h, d](f) \neq 0$  and (12) is false.

The next consequences follow immediately from Theorem 1.

**COROLLARY 1.** *Let  $A$  be a Banach algebra. On the hypothesis of Theorem 1 the spectral radius of  $d(x)$  vanishes.*

**COROLLARY 2** (see [2]). *Let  $A$  be a Banach algebra with unit and let  $d$  be a continuous derivation on  $A$ . If  $x \in A$  and  $[x, d(x)] = 0$  then*

$$\|(d(x))^n\|^{1/n} = 0 \left( \frac{1}{n} \right).$$

**COROLLARY 3** (see [1], [2]). *If  $d$  is a continuous derivation on a commutative Banach algebra  $A$  with unit then  $d(A) \subset \text{rad}(A)$ .*

Now we prove a generalization of the well-known Lars Hörmander Lemma (see [3] Lemma 4.1.1).

**THEOREM 2.** *Let  $E$  be a normed space with dual  $E'$  and let  $F$  be a complete separated convex space with dual  $F'$ . Let the strong topology of  $F'$  be determined by the family  $P$  of seminorms. Let  $t: E \rightarrow F$  be a linear mapping with the conjugate  $t'$ . We define:*

$$V = \{ \varphi \in F' \text{ such that } \varphi \circ t \in E' \}$$

and  $T = t' | V$ . If  $V$  is dense in  $F'$  under the  $\sigma(F', F)$  topology then the following conditions are equivalent:

- (i)  $T$  is an injection and  $T^{-1}$  is continuous under the topologies  $\sigma(E', E)$  and  $\sigma(F', F)$ ,
- (ii)  $t(E) = F$ ,
- (iii) for each  $p \in P$  there is  $\alpha > 0$  such that  $p(\varphi) \leq \alpha \|T(\varphi)\|$  for all  $\varphi \in V$ .

Proof. (i)  $\Rightarrow$  (ii). Let  $y \in F$  and let  $G = T(V)$ . We define the linear functional  $L$  on  $G$  as  $L: G \ni T(\varphi) \mapsto \varphi(y) \in K$ . This linear form is obviously continuous under the  $\sigma(E', E)$  topology. By the Hahn-Banach Theorem  $L$  has  $\sigma(E', E)$ -continuous extension to  $E'$ . On the other hand the mapping

$$(E, \sigma(E, E')) \ni u \mapsto (E' \ni \varphi \mapsto \varphi(u) \in K) \in (E', \sigma(E', E))'$$

is a linear homeomorphism. Hence there is  $x \in E$  such that  $T(\varphi)x = \varphi(y)$  for all  $\varphi \in V$ . Therefore  $\varphi(t(x)) = \varphi(y)$  for every  $\varphi \in F'$  (since  $\bar{V} = F'$ ). Thus  $t(x) = y$ .

(ii)  $\Rightarrow$  (iii). We define  $\Omega = T^{-1}(\{\psi \in E': \|\psi\| \leq 1\})$ . We now prove that  $\Omega$  is strongly bounded. It is enough to show that  $\Omega$  is  $\sigma(F', F)$  bounded (because  $F$  is a complete separated convex space). The set  $\{\varphi(y): \varphi \in \Omega\}$  is bounded for every  $y \in F$  since by (ii) there exists an element  $x$  of  $E$  such that  $t(x) = y$  and  $|\varphi(y)| = |\varphi(t(x))| = |T(\varphi)x| \leq \|x\|$ . Hence  $\Omega$  is  $\sigma(F', F)$  bounded and so strongly bounded.

(iii)  $\Rightarrow$  (i) is obvious.

Suppose now that  $A$  is a commutative Banach algebra with unit and let  $d$  be a  $(f, g)$ -derivation on  $A$  with the property  $d(\text{rad}(A)) \subset \text{rad}(A)$ . As an immediate conclusion from Theorem 2 we obtain some necessary and sufficient conditions for an equality  $d(\text{rad}(A)) = \text{rad}(A)$  to be true. (We define  $E = \text{rad}(A) = F$  and  $t = d$ ).

We finish this paper with a slight generalization of some results of Gorin, Kotchetkov and Mitiagin.

**THEOREM 3.** *Let  $A$  be a commutative Banach algebra with unit and let  $d$  be a continuous derivation on type  $(f, g)$  with the properties (7), (8), (9), (10), (11). If  $d(\text{rad}(A)) = \text{rad}(A)$  then there is  $c > 0$  such that:*

$$\chi_n \leq \exp\{-cn \text{Log}^2 n\}$$

where  $\chi_n = \sup\{\|x^n\|: x \in \text{rad}(A) \text{ and } \|x\| \leq 1\}$ .

We omit the proof because it is very similar to the proof in the special case ( $f(x) = g(x) = x$  for all  $x \in A$ ) as is shown in article [2].

Finally, I wish to thank Professor Andrzej Pelczar for his remarks.

#### References

- [1] I. M. Singer, J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. 129 (1955).
- [2] Е. А. Горин, Ю. Ю. Кочетков, Б. С. Митягин, *Некоторые свойства дифференцирований в банаховых алгебрах*, Теория Функций. Функциональный анализ и их приложения, Вып. 18, Харьков 1973.
- [3] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, New York 1966.
- [4] A. P. Robertson, W. J. Robertson, *Topological Vector Spaces*, Cambridge 1973.