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# Some properties of extremal functions

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*Abstract.* The object of this paper is to give some properties of the Leja-Siciak extremal function $\Phi(z, E)$ where $E$ is a compact balanced subset of $C^n$.

**1. Introduction.** The main result of this paper is a construction of an example of the discontinuous extremal function of closure of the polynomially convex domain of $C^n$, $n \geqslant 2$. This is the essential difference between an extremal function of $E \subset C$ and an extremal function of $E \subset C^n$, $n \geqslant 2$, because if $E$ is a plane set of positive capacity then the function $\Phi(z, E)$ is continuous in $\setminus E$, but for $E \subset C^n$, $n \geqslant 2$ it may be discontinuous at points of $\setminus E$.

Among other things the necessary and sufficient condition for the continuity of the extremal function for some class of circular compacts will be proved in this paper.

**2. Notations, definitions and some basic properties of the extremal function.** Let $C^n$ be a Cartesian product of $n$ complex planes. If the set $E \subset C^n$ then $\bar{E}$, $\operatorname{int} E$, $\partial E$, $\setminus E$ denote the closure, the interior, the boundary and the complement of the set $E$ to $C^n$, respectively.

If a compact set $E$ belongs to the domain of the function $f \subset C^n \times C$, then

$$\|f\|_E = \sup\{|f(z)|:\ z \in E\}.$$

If a function $f \subset C^n \times R$ ($R$ is a real line) then

$$f^*(z) = \limsup_{z' \to z} f(z').$$

We denote by $f|_E$ the restriction of a function $f$ to a set $E$. We designate by
$H(C^n)$ — family of homogeneous polynomials in $C^n$,
$W(C^n)$ — family of polynomials in $C^n$,
$P(D)$ — family of plurisubharmonic functions on $D$.
If $z \in C^n$ then $\|z\| = (|z_1|^2 + \ldots + |z|^2)^{1/2}$

$$K(z_0, r) = \{z \in C^n:\ \|z - z_0\| < r\},\ z_0 \in C^n,\ r > 0.$$

We put

$$H(E) := \{p \in H(C^n):\ \|p\|_E \leqslant 1\},$$
$$W(E) := \{p \in W(C^n):\ \|p\|_E \leqslant 1\}.$$

A polynomially convex hull of a compact set $E \subset C^n$, is the set

$$\hat{E} = \{z \in C^n\colon\ |p(z)| \leqslant \|p\|_E, p \in W(C^n)\}.$$

A compact set $E$ is a polynomially convex set if and only if $E = \hat{E}$. By analogy we define a homogeneous polynomially convex hull, $\hat{\hat{E}}$ of a compact set $E \subset C^n$ and a homogeneous polynomially convex set.

Let $E$ be a compact set in the space $C^n$. The extremal function of the set $E$ is defined as follows

$$\Phi(z, E)\colon = \sup\{|p(z)|^{1/\deg p}, p \in W(E)\},\ \deg p \neq 0.$$

Among other things the following properties of extremal function are known ([5], [6]).

Property 2.1. $\Phi(z, E)$ is lower semicontinuous.

Property 2.2. For any $z \in C^n$ and for any $p \in W(C^n)$

$$|p(z)| \leqslant \|p\|_E \cdot \Phi^{\deg p}_{(z,E)}.$$

Property 2.3. If $\Phi(z, E)$ is locally bounded, then

a) $\log \Phi^*(z, E) \in P(C^n)$,

b) $0 < \limsup\limits_{\|z\| \to \infty} \dfrac{\Phi(z, E)}{\|z\|}$.

3. The extremal function of circular sets. A set $E \subset C^n$ is circular if and only if the following implication is fulfilled

$$z_0 = (z_0^1, \ldots, z_0^n) \in E \Rightarrow e^{i\theta} z_0 = (e^{i\theta} z_0^1, \ldots, e^{i\theta} z_0^n) \in E \quad \text{for } 0 \leqslant \theta \leqslant 2\pi.$$

The extremal function of circular sets is equal to

(3.1) $\max\{1, \Psi(z, E)\}$ where $\Psi(z, E)\colon = \sup\{|p(z)|^{1/\deg p}\colon p \in H(E)\}$ ([5]).

THEOREM 3.1. *Let $D \subset C^n$ be a circular bounded domain of holomorphy defined by a plurisubharmonic positive homogeneous function $g$ such that*

$$D = \{z \in C^n\colon\ g(z) < 1\},$$

*then the following conditions are equivalent*:

(i) *there exists a family $F \subset H(C^n)$ such that $D = \operatorname{int} E$, where*

$$E = \{z \in C^n\colon\ |p(z)| \leqslant 1,\ p \in F\},$$

(ii) *there exists a sequence of Weils polyhedrons $\{E_\nu\}$ (for def. see* [7]) *defined by homogeneous polynomials such that $E_\nu \supset E_{\nu+1} \supset \ldots \supset D$*, int $\bigcap\limits_{\nu=1}^{\infty} E_\nu = D$,

(iii) *$D$ has no outer envelope of holomorphy* [*for def. see* [1]),

(iv) $\Psi^*(z, \bar{D}) = g(z)$.

**Proof:** (i)⇒(ii). Let $U_\nu = \bigcup_{a\in E} K\left(a, \frac{1}{\nu}\right)$. There exists a finite sequence $\{p_s\}\subset F$, $s = 1, \ldots, k_\nu$ such that

$$\max_{s=1\cdots k_\nu} |p_s(\zeta)|>1 \quad \text{for} \quad \zeta\in\partial U_\nu .$$

The sequence of polyhedrons $\{E_\nu\}$ defined by:

$$E_\nu := \{z\in C^n:\ \max_{s=1\cdots k_\nu} |p_s(z)|\leqslant 1\}\cap E_{\nu-1},\ \nu\geqslant 2$$

where

$$E_1 := \{z\in C^n:\ \max_{s=1\cdots k_1} |p_s(z)|\leqslant 1\}$$

fulfills the conditions of (ii).

(ii)⇒(iii). Let $z_0\in \bigcap_{\nu=1}^{\infty} E_\nu$. There exists $\nu_0$ such that $z_0\notin E_{\nu_0}$. Hence there exists $Q\in H(C^n)$ such that $\|Q\|_E\leqslant 1$ and $|Q(z_0)|>1$. The function $[Q(z)-Q(z_0)]^{-1}$ is holomorphic on $\bar{D}\subset \bigcap_{\nu=1}^{\infty} E_\nu$ and the same function is not holomorphic in $z_0$.

If $D'$ is the outer envelope of holomorphy of $D$ then we have

$$D\subset D'\subset \operatorname{int}\bigcap_{\nu=1}^{\infty} E_\nu = D .$$

(iii)⇒(iv). Not losing the generality we assume that $g$ is a homogeneous function of the first order.

We take any line $\{z\in C^n:\ z = ta,\ t>0,\ \|a\| = 1,\ a\in C^n\}$ and $z_0$ the first boundary point of $D$ on this line. We have

$$\Psi^*(z_0, \bar{D})\leqslant 1 = g(z_0) .$$

$$\text{So } \Psi^*(z, \bar{D})\leqslant g(z) \quad \text{for} \quad z\in C^n .$$

If for some $z_1\in C^n$ $\Psi^*(z_1, \bar{D})<g(z_1)$ then the domain

$$D' = \{z\in C^n:\ \Psi^*(z, \bar{D})<1\}\supsetneqq D .$$

Let $f$ be a holomorphic function on $\bar{D}$. Then there exists a series of homogeneous polynomials such that

$$f(z) = \sum_{\nu=1}^{\infty} Q_\nu(z), \qquad Q_\nu\in H(C^n), \qquad \nu = \deg Q_\nu$$

and this series is uniformly convergent in $\bar{D}$. Because

$$|Q_\nu(z)|\leqslant \|Q_\nu\|_{\bar{D}}\cdot \Psi^\nu(z, \bar{D}) \qquad \text{for } z\in C^n$$

so this function is continuable to the holomorphic function on $D'$. Hence

$$D\subsetneqq D'\subset D .$$

This contradiction proves that $\Psi^*(z, \bar{D}) = g(z)$.

(iv)⇒(i). The family of the extremal homogeneous polynomials of the set $\bar{D}$ (see [5]) fulfils the required conditions.

THEOREM 3.2. *Let $E \subset C^n$ be a compact circular set and*

$$\pi_a := \{z \in C^n : z = \lambda a, a \in C^n, \lambda \in C\} \qquad \|a\| \neq 1 .$$

*Then $\Phi(z, E)$ is continuous if and only if $\partial \hat{\hat{E}} \cap \pi_a$ is a circle for any $a$* (compare [3]).

Proof. Let $\partial \hat{\hat{E}} \cap \pi_a$ is a circle for any $a \in C^n$ with $\|a\| = 1$. By (3.1) it suffices to show that $\Psi(z, E)$ is continuous. Obviously

$$\Psi(z, E) = \Psi(z, \hat{\hat{E}})$$

so

$$\operatorname{int} \hat{\hat{E}} = \{z \in C^n : \Psi^*(z, E) < 1\} = \{z \in C^n : \Psi(z, E) < 1\} .$$

Because $\Psi(z, E)$ and $\Psi^*(z, E)$ are absolutely homogeneous of the first order so

$$\Psi(z, E) = \Psi^*(z, E) \qquad \text{for } z \in C^n$$

and this implies the continuity of $\Psi(z, E)$. If $\Phi(z, E)$ is continuous in $C^n$ then $\Psi(z, E)$ is continuous as well by (3.1) hence $\partial \hat{\hat{E}} = \{z \in C^n : \Psi(z, E) = 1\}$ and this implies the required condition.

THEOREM 3.3. *If a domain $D$ fulfils the conditions of Theorem* 3.1 *and $\{E_\nu\}$ is the sequence of compact sets such that $\bigcup_{\nu=1}^{\infty} E_\nu = D$, $E_\nu \subset E_{\nu+1} \ldots \subset D$, $\nu = 1, 2 \ldots$ then*

$$\lim_{\nu \to \infty} \Phi(z, E_\nu) = \Phi^*(z, \bar{D}) .$$

Proof. Because $\Psi^*(z, \bar{D})$ is a plurisubharmonic function absolutely homogeneous of first order so there exists a sequence of functions $\{g_\nu\} \subset P(C^n)$ such that $g_\nu$ are continuous absolutely homogeneous of first order for $\nu = 1, 2, \ldots$, and

$$g_\nu \searrow \Psi^*(z, \bar{D}) \ ([4]), \ g_\nu(z) > g_{\nu+1}(z) \quad \textit{for } z \in C^n \setminus \{0\} .$$

Let $F_\nu$ be the set defined as follows

$$F_\nu = \{z \in C^n : g_\nu(z) \leqslant 1\}$$

so

$$\Psi(z, F_\nu) = g_\nu(z), \ F_\nu \subset F_{\nu+1} \subset \ldots \subset D, \ \bigcup_{\nu=1}^{\infty} F_\nu = D .$$

Then for any $\nu$ there exist $k_\nu$ and $l_{k_\nu}$ such that

$$F_\nu \subset E_{k_\nu} \subset F_{l_{k_\nu}}$$

hence

$$\Phi(z, F_\nu) \geqslant \Phi(z, E_{k_\nu}) \geqslant \Phi(z, F_{l_{k_\nu}}) .$$

If $\nu$ tends to infinity we have

$$\Phi^*(z, \bar{D}) = \max\{1, \Psi^*(z, \bar{D})\} = \lim_{\nu\to\infty}\max\{1, \Psi(z, F_\nu)\}$$
$$\geqslant \lim_{\nu\to\infty} \Phi(z, E_{k_\nu}) \geqslant \lim_{\nu\to\infty}\max\{1, \Psi(z, F_{l_{k_\nu}})\} = \Phi^*(z, \bar{D}) .$$

Because $\{\Phi(z, E_\nu)\}$ is the convergent sequence so the limit of any subsequence of it is equal to the limit of $\{\Phi(z, E_\nu)\}$.

THEOREM 3.4. *Let $E \subset C^n$ be a circular compact set such that its extremal function is locally bounded and $\{E_\nu\}$ be a sequence of closed circular sets such that*

$$E_\nu \subset E_{\nu+1} \subset \ldots \subset E, \ \bigcup_{\nu=1}^{\infty} E_\nu = \hat{\hat{E}},$$

*then*

$$\lim_{\nu\to\infty} \Phi(z, E_\nu) = \Phi(z, E) .$$

Proof. Assume that there exists $z_0 \in C^n$ such that

$$\lim_{\nu\to\infty} \Psi(z_0, E_\nu) > \Psi(z_0, E)$$

then

$$\Psi(z_0, E_\nu) > \Psi(z_0, E) \quad \text{for } \nu = 1, 2, \ldots$$

Hence there exists $\lambda_0 > 0$ such that

$$\Psi(\lambda_0 z_0, E_\nu) \geqslant \lim_{\nu\to\infty} \Psi(\lambda_0 z_0, E_\nu) = 1 > \Psi(\lambda_0 z_0, E) \quad \text{for } \nu = 1, 2, \ldots$$

This implies for $\lambda > \lambda_0$ that $\lambda z_0 \notin E_\nu$ for $\nu = 1, 2, \ldots$ so $\lambda z_0 \notin \bigcup_{\nu=1}^{\infty} E_\nu$ and $\lambda z_0 \in E$ if $\lambda$ is sufficiently close to $\lambda_0$, hence $E \underline{\not\supseteq} \bigcup_{\nu=1}^{\infty} E_\nu$. This contradiction proves that

$$\lim_{\nu=\infty} \Phi(z, E_\nu) = \Phi(z, E) .$$

**4.** An example of a domain of holomorphy such that the extremal function of its closure is discontinuous.

THEOREM 4.1. *Let $E \subset C^n$ be a compact set such that its extremal function $\Phi(t, E)$ is locally bounded and let $\beta_1, \ldots, \beta_{n+1}, \alpha, R$ be a sequence of numbers which satisfies the following conditions*

$$\sum_{i=1}^{n+1} \beta_i = 1, \ \beta_i \geqslant 0, \ i = 1, \ldots, n+1, \ 0 < \alpha < \beta_{n+1}, \ R > 0 .$$

*Let*

$$g'(z) = \begin{cases} |z|^{\beta}\Phi^{\alpha}\left(\frac{z}{z_{n+1}}\right), & z_{n+1} \neq 0, \left(\frac{z_1}{z_{n+1}}, \dots \frac{z_n}{z_{n+1}}\right) = \left(\frac{z}{z_{n+1}}\right) \\ 0 & , z_{n+1} = 0, |z|^{\beta} = |z_1|^{\beta_1} \dots |z_{n+1}|^{\beta_{n+1}} \end{cases}$$

*Let moreover* $D \subset C^{n+1}$ *be a domain defined by*

$$D = \{z \in C^{n+1}: g^*(z) < 1\}$$

*where*

$$g(z) = \max\{R\|z\|, g'(z)\}.$$

*Then*

(i) *D is a bounded circular domain of holomorphy,*

(ii) $D = (\bar{D})^0$,

(iii) $\Psi^*(z, \bar{D}) = g^*(z)$.

Without loss of generality we prove Theorem 4.1 in the following particular case.

THEOREM 4.1′. *Let* $E \subset C$ *be a compact set such that its extremal function* $\Phi(t, E)$ *is locally bounded and let the domain D be defined by*

$$D = \{z \in C^2: g^*(z) < 1\}$$

*where*

$$g(z) = \max\{R\|z\|, g'(z)\} \text{ and}$$

$$g'(z) = \begin{cases} |z_2| \cdot \Phi^{1/2}\left(\frac{z_1}{z_2}\right), & z_2 \neq 0 \\ 0 & , z_2 = 0 \end{cases}$$

*Then*

(i) *D is a circular bounded domain of holomorphy,*

(ii) $D = (\bar{D})^0$,

(iii) $\Psi^*(z, \bar{D}) = g^*(z)$.

Proof. The boundedness of $D$ is obvious, so to have (i) it suffices to prove that $g^*(z)$ is plurisubharmonic, absolutely homogeneous of first order function. The absolute homogeneity follows immediately from the definition. It is plurisubharmonic outside the plane $z_2 = 0$ ([4]). We take any $z_0 = (z_0^1, 0)$ and any sequence $\{z_k\}$ $k = 1, 2, \dots$ such that $z_k \to z_0$. We may accept that the sequence

$$\left\{\frac{|z_k^1|}{|z_k^2|}\right\} \tag{4.1}$$

tends to some finite number or to infinity because when it is necessary we may consider the subsequences of sequence (4.1). We have the following alternative estimations:

$$0 \leqslant \lim_{z_k \to z_0} g'(z_k) = \lim_{z_k \to z_0} (|z_k^2| \cdot |z_k^1|)^{1/2} \left(\frac{\Phi\left(\frac{z_k^1}{z_k^2}\right)}{\left|\frac{z_k^1}{z_k^2}\right|}\right)^{1/2} \leqslant 2d \lim_{z_k \to z_0} (|z_k^2| \cdot |z_k^1|)^{1/2} = 0,$$

where the constant $d>0$ depends on $E$ (Property 2.3), or

$$0 \leqslant \lim_{z_k \to z_0} g'(z_k) = \lim_{z_k \to z_0} |z_k^2| \cdot \Phi^{1/2}\left(\frac{z_k^1}{z_k^2}\right) \leqslant A \cdot \lim_{z_k \to z_0} |z_k^2| = 0\,,$$

where the constant $A$ depends on the limit of sequence (4.1). This fact implies that $g'(z)$ is continuous in $z_0$, so

$$g'^*(z_0) = 0 = g'(z_0)\,.$$

Hence $g'^*(z)$ is plurisubharmonic in $C^2$ because

$$0 = g'^*(z_0) \leqslant \frac{1}{V} \int\limits_{K(z_0,\varrho)} g'^*(z)\,dv \quad \text{for} \quad \varrho>0$$

so $g^*(z) = \max\{R\|z\|, g'^*(z)\}$ is plurisubharmonic function in $C^2$, also.

In order to prove (iii) we construct a sequence of polyhedrons which satisfies the conditions (ii) of Theorem 3.1.

Let $\{L_i^\nu\}$, $\nu = 1, 2, \ldots$, $i = 1, 2, \ldots, \nu^*$ be the sequence of extremal polynomials of the set $E$. We construct the sequence of functions

$$g_\nu(z) = \max\{e^{-1/\nu} \cdot R\|z\|, g_\nu'(z)\}, \text{ where}$$

$$g_\nu'(z) = \begin{cases} e^{-1/\nu}|z_2| \max\limits_{i=1\ldots\nu^*} \left|L_i^\nu\left(\frac{z_1}{z_2}\right)\right|^{1/2\nu}, & z_2 \neq 0 \\ 0, & z_2 = 0\,. \end{cases}$$

Because the functions $g_\nu$, $\nu = 1, 2, \ldots$ are continuous and

$$g_\nu(z) < g_{\nu+1}(z) < \ldots < g(z) \quad \text{and} \quad g_\nu(z) \nearrow g(z)$$

we may take the sequence $\{E_\nu\}$ of polyhedrons defined by homogeneous polynomials such that

$$D_\nu \supset E_\nu \supset \hat{\hat{\bar{D}}}_{\nu+1} \supset \bar{D}_{\nu+1} \quad \text{for any } \nu \in N\,,$$

where

$$D_\nu = \{z \in C^2\colon\ g_\nu(z) < 1\}\,.$$

Moreover

$$E_\nu \supset E_{\nu+1} \supset \ldots \supset D \text{ and } \operatorname{int} \bigcap_{\nu=1}^{\infty} E_\nu = \operatorname{int} \bigcap_{\nu=1}^{\infty} \bar{D}_\nu\,.$$

If $z_0 \in \operatorname{int} \bigcap\limits_{\nu=1}^{\infty} E_\nu$ then there exists $\varrho>0$ such that

$$K(z_0, \varrho) \subset \operatorname{int} \bigcap_{\nu=1}^{\infty} E_\nu\,.$$

Hence we have the following sequence of implications

$$g_\nu(z)|_{K(z_0,\varrho)} \leqslant 1 \Rightarrow g(z)|_{K(z_0,\varrho)} \leqslant 1 \Rightarrow g^*(z)|_{K\left(z_0, \frac{\varrho}{2}\right)} \leqslant 1 \Rightarrow g^*(z_0) < 1 \Rightarrow z_0 \in D\,,$$

so the sequence $\{E_\nu\}$ satisfies the required conditions. Because the conditions (ii) and (iv) in Theorem 3.1 are equivalent, we have proved (iii).

(ii) is obvious because $D = \operatorname{int} \bigcap_{\nu=1}^{\infty} E_\nu$. Q.E.D.

Notice. If $\Phi^*(t, E)$ is discontinuous then this construction of the circular domain of holomorphy $D$ gives:

(i) the homogeneous polynomially convex domain of holomorphy such that the extremal function of its closure is discontinuous;

(ii) the homogeneous polynomially convex compact $(\hat{\bar{D}})$ such that its extremal function is discontinuous in some unbounded set $A$ and $A \cap (\setminus \hat{\bar{D}})$ is non-empty. This is impossible in the case when $n = 1$.

This notice gives a negative answer to question 2 in [8].

For example the extremal function of the set $E$ defined by $E = \{0\} \cup \sum_{j=1}^{\infty} E_j$, where $E_j = \{z \in C\colon |z - q^j| < r_j\}$, $0 < q < 1$, $r_j > 0$ is discontinuous if and only if

$$\sum_{j=1}^{\infty} j \ln 1\, r_j < \infty \quad ([2]).$$

Moreover $\Phi^*(z, E)$ is also a discontinuous function ([5]).

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