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DII PRACE MATEMATYCZNE, Z. 20 1979

Sufficiency of jets by the method of integral inequalities

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1. The theory of sufficiency of jets has now been essentially developed and many results have been published. The classical results are due to M. Morse [5, 6] (for instance the classical Morse lemma for homogeneous non-degenerate quadratic functions), N. H. Kuiper [9, 10], T. C. Kuo [11], J. Bochnak and S. Łojasiewicz [2]. Further results have been obtained by J. Bochnak [1], G. K. Francis [3], A. Ostrowski [7], R. Palais [8] and F. Takens [13]. For a general presentation of the Morse theory containing also some observations on the sufficiency of jets, we refer to the book by J. Milnor [4]. Further references may be found in the book by J. Cl. Tougeron [14], where some deep aspects of sufficiency with respect to the theory of C^∞-functions are discussed. Some special questions of the general theory of sufficiency are problems concerning the C^p-sufficiency of r-jets in C^r; these problems are discussed in particular in [12], [9], [11], [13]. The purpose of the present paper is to prove some results on the sufficiency of r-jets in C^r and C^{r+1}, by using the method of integral inequalities. We have no essentially new important results, but rather give a method (which seems to be very natural) of proving known theorems and some modifications of these giving slightly better results; a little formal progress lies in the fact that we can show that certain mappings are of a higher class of regularity than that which may be explicitly obtained by the other methods. More precisely, we shall show that a mapping constructed by T. C. Kuo [11] (in order to show that an r-jet fulfilling the "gradient inequality" (see (2) in Sec. 3 below) in C^0-sufficient in C^r) is differentiable at the origin (that is, it belongs to $C^{(d)}$; see Sec. 2 below), and not only continuous, as was proved in [11]. Note here that in Kuiper's paper [9] it was proved implicitly that a mappings (constructed by using the same idea as in [11]) which fulfils the C^0-sufficiency of an r-jet in C^r is of the class $C^{(d)}$, though this had not been explicitly stated in the corresponding theorems. It seems that $C^{(d)}$-sufficiency is useful; we refer to the last section of the present paper, where in a very simple situation the class of $C^{(d)}$ is essentially needed.

Our method may also be applied to prove results concerning $C^{1+(d)^0}$ and C^2-sufficiency of r-jets in C^{r+1} (if the appropriate conditions are satisfied). More precisely, we shall show that an r-jet fulfilling the "gradient inequality" and some additional conditions is C^2-sufficient in C^{r+1} and moreover the mapping, giving the C^2-equivalence of every two realizations of this jet, has the second differential vanishing at the origin. This statement has not been presented explicitly in other papers; it seems, however, that this fact might be useful in certain calculations. In Sec. 7 we give some observations on the $C^{(d)}$-sufficiency of r-jets in C^{r+1} on the assumption that they satisfy a weaker form of the "gradient inequality". In the last section we shall show the C^1-sufficiency of r-jets in C^r in a one-dimensional case.

It is clear that the method of integral inequalities may be extended in a natural manner to prove a higher class of sufficiency.

2. We recall the fundamental definitions, usual notation and some results, and also introduce some new definitions. An *r-jet* of a C^r-real function defined in a neighbourhood of $0 \in R^n$ such that $f(0) = 0$, is identified with the r-th Taylor polynomial at 0 of the function f; the function f is called a *realization* of the jet (see for instance [2], [9], [11]). We say that an r-jet is *C^p-sufficient in C^k* $(r, p, k = 0, 1, 2, \ldots)$ if and only if for any two of its realizations λ and μ, there is a (local) C^p-isomorphism $g\colon (Nbh\ 0, 0) \to (Nbh\ 0, 0)$ such that

(0) $\lambda \circ g = \mu$ in some neighbourhood (Nbh 0) of the origin, (see for example [2], [9] [11]).

We say that an r-jet is *$C^{(d)}$-sufficient in C^k* if and only if for any two of its realizations λ and μ there exists a differentiable (local) homeomorphism $g\colon (Nbh\ 0, 0) \to (Nbh\ 0, 0)$ having at the point zero the first differential equal to the identity mapping ($d_0 g = Id_{R^n}$), such that (0) holds true; an r-jet is *$C^{1+(d)^0}$-sufficient in C^k* if and only if for any two of its realizations λ and μ, there is a (local) C^1-diffeomorphism $g\colon (Nbh\ 0, 0) \to (Nbh\ 0, 0)$ such that there exists the second differential $d_0^2 g$ of the mapping g at the origin and this differential is equal to zero and moreover the first differential $d_0 g$ of g at the origin is the identity mapping.

The fundamental results (see [2], [9], [11]) on the C^0-sufficiency in C^r of r-jets give in particular the following

THEOREM 0. *An r-jet v is C^0-sufficient in C^r if and only if there exist two positive constants c and η^0 such that the following condition*

$$|(\operatorname{grad} v)(x)| \geqslant c|x|^{r-1} \quad \text{for } x \text{ such that } |x| \leqslant \eta^0, \tag{1}$$

called "gradient inequality", holds true; here $|x|$ denotes of course the norm of $x = (x_1, \ldots, x_n) \in R^n$.

It is clear that in (1) we can replace v by any realization f of it.

3. Let $v\colon R^n \to R$ be a polynomial of degree not greater than r, such that $v(0) = 0$, fulfilling (1) for some constants c and η^0. Let furthermore $P\colon R^n \to R$ be a given C^r-function such that $lim|P(x)|/|x|^r = 0$ as $x \to 0$.

Let us put

$$F(x, a) \overset{\mathrm{df}}{=} v(x) + a \cdot P(x) \quad \text{for} \quad a \in R,\ x \in R^n. \tag{2}$$

It may be shown (cf. Kuo [11], Lemma 1, p. 168) that for any $\varepsilon>0$ there is a number $\delta_\varepsilon>0$ such that

$$|(\operatorname{grad} F)(x, a)| \geqslant |(\operatorname{grad} V)(x)+a(\operatorname{grad} P)(x)| \geqslant \frac{\varepsilon}{2}\cdot|x|^{r-1} \tag{3}$$

for $|x|<\delta_\varepsilon$ and $0\leqslant a\leqslant 1$. Let us consider some fixed $\varepsilon>0$ and $\delta=\delta_\varepsilon>0$ chosen for ε as above, and let us put

$$X(x, a) \overset{\mathrm{df}}{=} P(x)\cdot((\operatorname{grad} F)(x, a))\cdot|(\operatorname{grad} F)(x, a)|^{-2} \tag{4}$$

for $a\in R$, $0<|x|<\delta$.

Finally we define a vector field

$$W\colon R^n\times R\to R^n\times R$$

in such a way that

$$W_i(x, a) = -X_i(x, a) \quad \text{for} \quad 0<|x|<\delta,\ a\in R,\ i=1,\dots,n, \tag{5}$$

$$W_{n+1}(x, a) = 1-X_{n+1}(x, a) \quad \text{for} \quad 0<|x|<\delta,\ a\in R, \tag{6}$$

$$W_i(0, a) = 0 \quad \text{for} \quad a\in R,\ i=1,\dots,n, \tag{7}$$

$$W_{n+1}(0, a) = 1 \quad \text{for} \quad a\in R; \tag{8}$$

(see Kuo [11]; compare also Kuiper [9] and Takens [13]). Putting $y=(x, a)\in R^n\times R$ we denote by

$$\varphi(\cdot\,; x)\colon t\mapsto\varphi(t; x)$$

the (unique) solution of the Cauchy problem

$$dy/dt = W(y) \tag{9}$$

$$y(0) = (x, 0)\ \ \big(y_i(0) = x_i \text{ for } i=1,\dots,n,\ y_{n+1}(0)=0\big). \tag{10}$$

The existence and uniqueness of solutions of (9)—(10) for each $x\in R^n$ such that $|x|<\delta$ was proved by Kuo in [11]. Each solution $\varphi(\cdot\,; x)$ of (9)—(10) cuts (at least once) the hyperplane $a=1$ (see [11]).

Let us denote by p the natural projection of $R^n\times R$ onto R^n. For any $x\in R^n$ we put

$$g(x) \overset{\mathrm{df}}{=} p(\varphi(t_x; x)) \tag{11}$$

where

$$t_x \overset{\mathrm{df}}{=} \inf\{t\geqslant 0\colon \varphi(t; x) = 1\}. \tag{12}$$

In other words, $g(x)$ is the projection on R^n of that point $y\in R^n\times R$ at which $\varphi(\cdot\,; x)$ cuts the hyperplane $a=1$ the first time. T. C. Kuo has proved in [11] that the mapping

$$x\mapsto g(x) \tag{13}$$

is of the class C^r in the set

$$\{x\in R^n\colon 0<|x|<\delta\} \tag{14}$$

and it is continuous at the origin. This means that g is a local homeomorphism $(Nbh\,0,0)\rightarrow(Nbh\,0,0)$ (see Lemma 2 in [11]). For $x \neq 0$ $W(x,0)$ is tangent to the level surface $F = \text{const}$, passing through (x,a), and then

$$v(x) = F(x,0) = F(g(x),1) = v(g(x))+P(g(x)).$$

This proves Theorem 1 in [11], stating that v is C^0-sufficient.

4. We shall prove below the following

Theorem A. *Let us suppose that the conditions assumed above in Sec. 3 are satisfied. Then there exists the first differential $d_0 g$ of the mapping g defined by* (11), *at the point* 0, *and it is equal to the identity mapping*:

$$d_0 g = Id_{R^n}.$$

Proof. The conclusion of our theorem is equivalent to saying that

$$\lim_{\substack{x\to 0\\ x\neq 0}} \frac{|g(x)-x|}{|x|} = 0. \tag{15}$$

In the sequel let ε, $\delta = \eta^0$ and c be the constants considered above in Sec. 3; thus in particular

$$|(\operatorname{grad} v)(x)| \geqslant c|x|^{r-1} \quad \text{for} \quad 0<|x|<\delta,$$

$$|(\operatorname{grad} F)(x,a)| \geqslant \frac{\varepsilon}{2}|x|^{r-1} \quad \text{for} \quad 0<|x|<\delta.$$

We shall now consider all functions, systems of equations etc., only for $x\in R^n$ such that $0<|x|<\delta$.

It is well known that the problem of existence and uniqueness of solutions of (9)—(10) is for each x equivalent to the problem of existence and uniqueness of solutions of the following system of integral equations:

$$\begin{cases} y_i(t) = x_i + \int_0^t W_i(y_1(s),\ldots,y_{n+1}(s))\,ds, & i=1,\ldots,n\\ y_{n+1}(t) = \int_0^t W_{n+1}(y_1(s),\ldots,y_{n+1}(s))\,ds; \end{cases} \tag{16}$$

briefly, the problem (9)—(10) is equivalent to (16).
Using the notation introduced above (see Sec. 3), we have obviously for $\varphi(\cdot\,;x)$, being the unique solution of (9)—(10), the following system of equalities:

$$d\varphi_i(t;x)/dt = W_i(\varphi_1(t;x),\ldots,\varphi_{n+1}(t;x)), \quad i=1,\ldots,n+1, \tag{17}$$

and then (in virtue of the fact that $\varphi(0;x) = (x,0)$):

$$\begin{cases} \varphi_i(t;x) = x_i + \int_0^t W_i(\varphi_1(s;x),\ldots,\varphi_{n+1}(s;x))\,ds, & i=1,\ldots,n\\ \varphi_{n+1}(t;x) = \int_0^t W_{n+1}(\varphi_1(s;x),\ldots,\varphi_{n+1}(s;x))\,ds. \end{cases} \tag{18}$$

It is easy to see that

$$(19)\qquad g_i(x) = \varphi_i(t_x; x) \quad (i = 1, \dots, n)$$

(compare (11) and (12)) and then the relation (15) is equivalent to

$$(20)\qquad \lim_{\substack{x\to 0\\ x\neq 0}} \frac{1}{|x|} \int_0^t W_i(\varphi_1(s; x), \dots, \varphi_{n+1}(s; x))\,ds| = 0, \quad i = 1, \dots, n.$$

Denote by $t_x(a)$ this value of the time t, for which

$$\varphi_{n+1}(t; x) = a \text{ (first time)};$$

this means that in particular

$$t_x(1) = t_x$$

(see (12)).

Putting

$$(21)\qquad \psi_i(a; x) \overset{\mathrm{df}}{=} \varphi_i(t_x(a); x) \quad i = 1, \dots, n,$$

we obtain from (17)

$$(22)\qquad d\psi_i(a; x)/da = \frac{W_i(\psi_1(a; x), \dots, \psi_n(a; x), a)}{W_{n+1}(\psi_1(a; x), \dots, \psi_n(a; x), a)}, \quad i = 1, \dots, n,$$

for a belonging to the closed interval $[0, 1]$. This is obvious in virtue of the classical theorem on the differentiability of compositions of functions. The differentiability of $t_x(\cdot)$ is a simple consequence of the properties of $\varphi_{n+1}(\cdot; x)$ and the implicit function theorem; the good properties of $\varphi_{n+1}(\cdot; x)$ follow from the properties of the vector field X discussed with all details in [11]. From the definition (2) of the function F we obtain directly:

$$(23)\qquad (\mathrm{grad} F)(x, a) = \left(\dots, \frac{\partial v}{\partial x_i}(x) + a\frac{\partial P}{\partial x_i}(x), \dots, P(x)\right) = (\mathrm{grad}\, v + a\,\mathrm{grad} P)(x).$$

Hence

$$X(x, a) = |(\mathrm{grad} F)(x, a)|^{-2}\cdot P(x)\cdot(\mathrm{grad} F)(x, a)$$

$$= \left(\dots, \frac{P(x)\cdot\left(\frac{\partial v}{\partial x_i}(x) + a\frac{\partial P}{\partial x_i}(x)\right)}{b(x, a)}, \dots, \frac{[P(x)]^2}{b(x, a)}\right),$$

where

$$(25)\qquad b(x, a) = \sum_{j=1}^{n}\left[\frac{\partial v}{\partial x_j}(x) + a\frac{\partial P}{\partial x_j}(x)\right]^2 + [P(x)]^2.$$

Thus, for $x \neq 0$, we have

$$W_i(x, a) = -\frac{P(x)\left(\frac{\partial v}{\partial x_i}(x) + a\frac{\partial P}{\partial x_i}(x)\right)}{b(x, a)} \quad i = 1, \dots, n,$$

and

$$W_{n+1}(x,a)=\frac{b(x,a)-[P(x)]^2}{b(x,a)}.$$

Then finally we have

$$W_i(x,a)/W_{n+1}(x,a)=-P(x)\left[\frac{\partial v}{\partial x_i}(x)+a\frac{\partial P}{\partial x_i}(x)\right]\Big/\left[b(x,a)-[P(x)]^2\right].$$

Hence

$$\frac{1}{|x|}|W_i(x,a)/W_{n+1}(x,a)|\leqslant\frac{|P(x)|}{|x|}\cdot\frac{\left|\frac{\partial F}{\partial x_i}(x,a)\right|}{b(x,a)}$$

$$\leqslant\frac{|P(x)|}{|x|}\frac{1}{|(\operatorname{grad}F)(x,a)|}\leqslant\frac{|P(x)|}{|x|\,|(\operatorname{grad}v)(x)+a(\operatorname{grad}P)(x)|}.$$

Since inequality (3) holds true, we obtain from the above estimations

$$\frac{1}{|x|}|W_i(x,a)/W_{n+1}(x,a)|\leqslant\frac{|P(x)|}{|x|^{r-1}|x|\operatorname{const}}\leqslant\tilde{c}|P(x)|\,|x|^{-r} \tag{26}$$

for some $\tilde{c}>0$.

Since $|P(x)|\,|x|^{-r}$ tends to zero as $|x|\to 0$, we have

$$\lim\frac{1}{|x|}(W_i(x,a)/W_{n+1}(x,a))=0\quad\text{as } x\to 0,\ x\neq 0 \tag{27}$$

and the convergence is uniform with respect to a in the interval $[0,1]$.

Let us put for abbreviation

$$w_i(s,a)=W_i(\psi_1(s;x),\ldots,\psi_n(s;x),s)/W_{n+1}(\psi_1(s;x),\ldots,\psi_n(s;x),s) \tag{28}$$

for $i=1,\ldots,n$.

Using the notation (28) we obtain:

$$\psi_i(a;x)=x_i+\int_0^a w_i(s,a)\,ds\quad(i=1,\ldots,n). \tag{29}$$

On the other hand we have

$$\psi(1;x)=g(x). \tag{30}$$

Proposition 1. *$\psi(s;x)=0(|x|)$ for $0<|x|<\delta$.*

Proof. From (29) we obtain

$$\frac{1}{|x|}|\psi_i(a;x)|\leqslant 1+\int_0^a|w_i(s;x)|\frac{1}{|\psi(s;x)|}\cdot\frac{|\psi(s;x)|}{|x|}ds. \tag{31}$$

From (27) it follows that there exists a positive constant C_1 such that

$$|w_i(s,x)|/|\psi(s;x)|\leqslant C_1. \tag{32}$$

Hence, there exists a positive constant C_2 such that

$$\frac{|\psi(a;x)|}{|x|} \leqslant n + \int_0^a C_2 \frac{|\psi(s;x)|}{|x|}\,ds \quad \text{for } 0<|x|<\delta,\ 0\leqslant a\leqslant 1\,. \tag{33}$$

In virtue of well-known results from the theory of integral inequalities, we have

$$\frac{|\psi(a;x)|}{|x|} \leqslant n\cdot\exp(aC_2) \leqslant C_3 = \text{const}\,; \tag{34}$$

the constant C_3 does not depend on x and a.

The proof of Proposition 1 is completed.

We shall now consider

$$\frac{1}{|x|}|w_i(s,x)| = |w_i(s,x)|\cdot\frac{1}{|\psi(s;x)|}\cdot\frac{|\psi(s;x)|}{|x|}\,.$$

From the uniform convergence (with respect to s in $[0,1]$):

$$|\psi(s;x)|\to 0 \text{ as } |x|\to 0$$

and the relation (27), we easily obtain

$$\frac{1}{|x|}|w_i(s,x)|\to 0 \quad \text{as } |x|\to 0\,, \tag{35}$$

uniformly with respect to s in $[0,1]$. From (35) we can conclude immediately

$$\lim \frac{1}{|x|}|\psi(1;x)-x| = 0 \quad \text{as } x\to 0,\ x\neq 0\,, \tag{36}$$

because (see (29))

$$\frac{1}{|x|}|\psi_i(1;x)-x_i| \leqslant \int_0^1 |w_i(s,x)|\cdot\frac{1}{|x|}\,ds \quad \text{for} \quad 0<|x|<\delta\ (i=1,\dots,n)\,.$$

In virtue of (30), the relation (36) is equivalent to (15) and then the proof of Theorem A is finished.

As a *corollary* we obtain the following statement:

If an r-jet v is such that (1) *is satisfied, then v is $C^{(d)}$-sufficient in C^r.*

This statement is implicitly contained in Kuiper's paper [9].

Remark 0. In virtue of the main result of [2] we have the following obvious conclusion from the above statement: For an r-jet v the following conditions are equivalent:

(i) v fulfils (1);

(ii) v is C^0-sufficient in C^r;

(iii) v is $C^{(d)}$-sufficient in C^r.

5. Consider now the following problem: on which additional assumptions (or without any supplementary conditions?) it is true that

$$\lim(\partial g_i/\partial x_j)(x) = \delta_{ij} \text{ as } x \to 0\,. \tag{37}$$

This is a particular question taken from the following problem: whether it is true that the condition (1) is sufficient for an r-jet v to be C^1-sufficient in C^r. This problem was posed by Professor S. Łojasiewicz in 1970 during the Liverpool Singularities Symposium, and is still open. We shall assume that the function v is as in Sec. 3 and we shall use the notation introduced in Sec. 3.

We have

$$\frac{\partial\psi_i}{\partial x_j}(t;x) = \delta_{ij} + \int\limits_0^t \frac{\partial}{\partial x_j} w_i(s,x)\,ds\,, \qquad i,j = 1,\dots,n\,. \tag{38}$$

Using the notation (25) we may write:

$$\frac{\partial}{\partial u_k}(V_i/V_{n+1})(u_1,\dots,u_n,a) = \frac{\partial}{\partial u_k}\left\{-P(x)\left[\frac{\partial v}{\partial u_i}(u) + a\frac{\partial P}{\partial u_i}(u)\right]\Big/\left[b(u,a) - [P(u)]^2\right]\right\}$$

Putting

$$d(u,a) \overset{\text{df}}{=} [b(u,a) - (P(u))^2]^{\frac{1}{2}} = |(\operatorname{grad} v)(u) + a(\operatorname{grad} P)(u)| \tag{39}$$

we obtain

$$\begin{aligned}
&\frac{\partial}{\partial u_k}(V_i/V_{n+1})(u_1,\dots,u_n,a)\\
&\quad = -\frac{1}{d(u,a)}\cdot\frac{\partial P}{\partial u_k}(u)\cdot\frac{1}{d(u,a)}\cdot\left[\frac{\partial v}{\partial u_i}(u) + a\frac{\partial P}{\partial u_i}(u)\right] -\\
&\quad\quad -\frac{P(u)}{d(u,a)}\cdot\frac{1}{d(u,a)}\cdot\left[\frac{\partial^2 v}{\partial u_i\partial u_k}(u) + a\frac{\partial^2 P}{\partial u_i\partial u_k}(u)\right] +\\
&\quad\quad +2\frac{P(u)}{d(u,a)}\cdot\frac{1}{d(u,a)}\cdot\left[\frac{\partial v}{\partial u_i}(u) + a\frac{\partial P}{\partial u_i}(u)\right]\times\\
&\quad\quad \times\sum_{j=1}^{n}\frac{1}{d(u,a)}\left[\frac{\partial v}{\partial u_j}(u) + a\frac{\partial P}{\partial u_j}(u)\right]\cdot\frac{1}{d(u,a]}\cdot\left[\frac{\partial^2 v}{\partial u_j\partial u_k}(u) + a\frac{\partial^2 P}{\partial u_j\partial u_k}(u)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\frac{\partial}{\partial u_k}(V_i/V_{n+1})(u_1,\dots,u_n,a)\right| &\leqslant \frac{\left|\frac{\partial P}{\partial u_k}(u)\right|}{d(u,a)} + \frac{|P(u)|}{d(u,a)}\cdot\frac{\left|\frac{\partial^2 v}{\partial u_i\partial u_k}(u) + a\frac{\partial^2 P}{\partial u_i\partial u_k}(u)\right|}{d(u,a)} +\\
&\quad +2\frac{|P(u)|}{d(u,a)}\cdot\sum_{j=1}^{n}\frac{\left|\frac{\partial^2 v}{\partial u_j\partial u_k}(u) + a\frac{\partial^2 P}{\partial u_j\partial u_k}(u)\right|}{d(u,a)}
\end{aligned}$$

We have

$$d(u,a) = |(\operatorname{grad} v + a \operatorname{grad} P(u)| \geqslant \text{const} \cdot |u|^{r-1}$$

(see (39) and (3)). Moreover

$$P(x)/|x|^r \to 0 \quad \text{as } x \to 0,\ x \neq 0$$

and then

$$\left|\frac{\partial P}{\partial u_k}\right| \Big/ |u|^{r-1} \to 0 \qquad \text{as } u \to 0,\ u \neq 0\,.$$

Thus, in virtue of the inequality:

$$\left|\frac{\partial P}{\partial u_k}(u)\right| \Big/ |(\operatorname{grad} v)(u) + a(\operatorname{grad} P)(u)| \leqslant \text{const} \left|\frac{\partial P}{\partial u_k}(u)\right| \Big/ |u|^{r-1}\,,$$

we have

$$\left|\frac{\partial P}{\partial u_k}(u)\right| \Big/ |(\operatorname{grad} v)(u) + a(\operatorname{grad} P)(u)| \to 0 \qquad \text{as } u \to 0\,.$$

Also we have obviously

$$|P(u)|/|(\operatorname{grad} v)(u) + a(\operatorname{grad} P)(u)| \to 0 \text{ as } u \to 0\,.$$

Thus we have proved that a sufficient condition for the relation

$$\frac{\partial}{\partial u_k}(V_i/V_{n+1})(u,a) \to 0 \qquad \text{as } u \to 0;\ u \neq 0 \tag{40}$$

is the boundedness of

$$\frac{\left|\dfrac{\partial^2 v}{\partial u_i \partial u_k}(u) + a \dfrac{\partial^2 P}{\partial u_i \partial u_k}(u)\right| |u|}{\left\{\displaystyle\sum_{j=1}^{n} \left[\frac{\partial v}{\partial u_j}(u) + a \frac{\partial P}{\partial u_j}(u)\right]^2\right\}^{1/2}} \tag{41}$$

near zero, for every i and every k.

It is clear that, in virtue of the condition assumed for the function P, the boundedness of (41) is assured if

$$\frac{\left|\dfrac{\partial^2 v}{\partial u_i \partial u_k}(u)\right| |u|}{|(\operatorname{grad} v)(u)|} \leqslant \text{const}\,, \tag{42}$$

in a neighbourhood of 0. This is satisfied for instance if

$$v(x) = a_1 x_1^r + \ldots + a_n x_n^r\,.$$

Thus we have proved the following

PROPOSITION 2. *If an r-jet v satisfies* (1) *and* (42) *then* (40) *holds true.*

As a corollary of Proposition 2 we obtain the following

PROPOSITION 3. *Suppose the assumptions of Proposition 2. Then* (37) *holds true.*

Proof. We have

$$(43)\qquad \frac{\partial\psi_i}{\partial x_j}(t;x) = \delta_{ij}+\int\limits_0^t \frac{\partial}{\partial x_j}\left(\frac{V_i}{V_{n+1}}(\psi_1(s;x),\ldots,\psi_n(s;x),s)\right)ds$$

$$= \delta_{ij}+\int\limits_0^t \left\{\sum_{k=1}^{n}\left(\frac{\partial}{\partial u_k}\left(\frac{V_i}{V_{n+1}}(u_1,\ldots,u_n,s)\right|_{u_j=\psi_j(s;x)}\right)\frac{\partial\psi_k}{\partial x_j}(s;x)\right\}ds.$$

Since

$$\psi(s;x)\to 0 \text{ as } x\to 0, \text{ uniformly in } s,$$

we have

$$(44)\qquad \left(\frac{\partial}{\partial u_k}\left(\frac{V_i}{V_{n+1}}\right)\right)(u,s)\Bigg|_{u_j=\psi_j(s;x)}\to 0 \quad \text{as } x\to 0,\ x\neq 0$$

and the convergence is uniform with respect to s; this is a direct corollary of Proposition 2. In particular

$$\left|\frac{\partial}{\partial u_k}\left(\frac{V_i}{V_{n+1}}\right)\right|$$

with $u=\psi(s;x)$, are bounded for $i,k=1,\ldots,n$, $|x|<\delta$.
Hence, using the same reasoning as in the proof of Proposition 1, we can prove that

$$\frac{\partial\psi_i}{\partial x_j}(t;x) \qquad i,j=1,\ldots,n$$

are bounded for $|x|<\delta$. Thus, from (43) and (44) we obtain

$$\frac{\partial\psi_i}{\partial x_j}(t;x)\to\delta_{ij} \quad \text{for} \quad x\to 0,\ x\neq 0,\ i,j=1,\ldots,n.$$

In particular

$$\frac{\partial\psi_i}{\partial x_j}(1;x)=\frac{\partial g_i}{\partial x_j}(x)\to\delta_{ij} \quad \text{for} \quad x\to 0.$$

The proof of Proposition 3 is completed.

From Proposition 3 we easily obtain the following

THEOREM B. *If v is a polynomial of the degree not greater than r, and fulfils the conditions assumed in Sec.* 3 (*in particular the condition* (1)) *and the condition* (42), *then the function g defined in Sec.* 3 *by the formula* (11) *is of the class C^1 in a neighbourhood $\{x\in R^n\colon |x|<\delta\}$ of zero.*

Proof. First of all we recall that the function g is of the class C^r in every set $\{x\in R^n\colon 0<|x|<\delta\}$, (this was proved by Kuo [11]). Using Proposition 3 we have also smoothness of the function g at the origin: it must be of the class C^1. The proof is completed.

Remark 1. It is easy to see that the assumptions of Theorem B are fulfilled in particular for

$$v(x) = a_1 x_1^r + a_2 x_2^r + \dots + a_n x_n^r \tag{45}$$

where $a_i \neq 0$ for every i.

COROLLARY. *An r-jet v of the form* (45) *is C^1-sufficient in C^r.*

This result was proved by Kuiper [9] and Takens [13]; the results of Takens are more general, and so our Corollary is a special case of the main result of [13].

6. Now we shall show that our method may be successfully used for proving further results on the sufficiency of r-jets and $(r+q)$-jets; we may also in this way obtain a special case of the Takens result [13]. Moreover we can show explicitly that — for instance — a diffeomorphism which realizes the C^2-equivalence of our jet v, given previously, and a function $v+P$ (where P is of the class C^{r+1} and has the first, second..., $(r+1)$-th differential at the origin equal to zero) has the first differential at the origin equal to the identity mapping and the second differential at the origin equal to zero.

More precisely, we shall show the following

THEOREM C. *Suppose that the polynomial v of the degree not greater than r satisfies* (1), (42) *and*

$$|(\partial^3 v/\partial x_i \partial x_j \partial x_k)(x)|\,|x| \leqslant \mathrm{const}\cdot|(\operatorname{grad} v)(x)| \; ; \tag{46}$$

near the origin. Assume also that the function P considered previously fulfils the condition

$$P(x)/|x|^{r+1} \to 0 \text{ as } x \to 0,\ x \neq 0 \, . \tag{47}$$

Suppose that $r \geqslant 2$. Then the function g defined by (11) *is of the class C^2 in a neighbourhood of the origin, the first differential $d_0 g$ of g at* 0 *is the identity mapping, the second differential $d_0^2 g$ of g at* 0 *is equal to zero, and — obviously — the following identity*

$$v(x) \equiv (v+P)(x)$$

holds true in a neighbourhood of the origin.

Proof. Since g is of the class C^r in the set (14), it is sufficient to prove that $d_0^2 g$ exists and that $d_0^2 g$ is continuous at the point zero, and moreover $d_0^2 g = 0$.

I. We shall show that there exist partial derivatives

$$\frac{\partial^2 g_i}{\partial x_j \partial x_k}(0) \qquad i, j, k = 1, \dots, n \tag{48}$$

of the function g at the origin.

PROPOSITION 4. *If all assumptions of Theorem C are satisfied, then*

$$\lim_{\substack{x \to 0 \\ x \neq 0}} \frac{1}{|x|} \frac{\partial}{\partial u_k} \left(\frac{V_i}{V_{n+1}}(u_1, \dots, u_n, s) \right) \Bigg|_{u_j = \psi_j(s;x)} = 0 \tag{49}$$

and the limit is uniform in s (ψ is defined by (21), *V_j by* (5)—(8)).