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A homomorphismus of certain Stein algebras

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Abstract. An elementary proof of the following result is given.

Let V_1 and V_2 be analytic subvarieties of C^n and C^m respectively, and let $\mathcal{O}(V_1)$ and $\mathcal{O}(V_2)$ be the algebras of all holomorphic functions on V_1 and V_2. Let $h: \mathcal{O}(V_2) \to \mathcal{O}(V_1)$ be a homomorphism of the C-algebras. Then there exists a unique holomorphic map $f: V_1 \to V_2$ such that $h = f^*$, where $f^*: \mathcal{O}(V_2) \ni \varphi \to \varphi \circ f \in \mathcal{O}(V_1)$.

A more general result is given by O. Forster in [1]. Our proof depends only on the following Theorem (*) and on some properties of entire functions.

Theorem (* [2]). *Every homomorphism of C-algebras $\mathcal{O}(V_2)$ and $\mathcal{O}(V_1)$ is continuous, if we consider $\mathcal{O}(V_2)$ and $\mathcal{O}(V_1)$ as Frechet-algebras and V_2, V_1 as Stein spaces.*

Let V be a complex analytic subvariety of C^n, and $\mathcal{O}(V)$ denote the algebra of all restrictions of entire functions on C^n to V. Cartan's Theorem B implies immediately that every holomorphic function on V, as a function on an analytic subspace of C^n, is extendible to an entire function on C^n.

Analogously, every holomorphic map $f: V_1 \to V_2$ is extendible to a holomorphic map $\tilde{f}: C^n \to C^m$, if V_1 and V_2 are analytic sets in C^n and C^m respectively.

Every holomorphic map $f: V_1 \to V_2$ induces in a natural manner the homomorphism f^* of $\mathcal{O}(V_2)$ into $\mathcal{O}(V_1)$.

Let Hol (V_1, V_2) denote the set of all holomorphic maps from V_1 to V_2 and let $\mathrm{Hom}_{C\text{-alg}}(\mathcal{O}(V_2), \mathcal{O}(V_1))$ denote the set of all C-algebra homomorphisms of $\mathcal{O}(V_2)$ into $\mathcal{O}(V_1)$.

Now we are able to prove the following

Theorem. *The mapping Hol $(V_1, V_2) \ni f \to f^* \in \mathrm{Hom}_{C\text{-alg}}(\mathcal{O}(V_2), \mathcal{O}(V_1))$ is bijective.*

Proof. Injectivity. Let $f_1, f_2 \in \mathrm{Hol}(V_1, V_2)$, $f_1 \neq f_2$. Then there exists $a \in V_1$ such that $f_1(a) \neq f_2(a)$. Since $\mathcal{O}(V_2)$ separates the points, there exists $\varphi \in \mathcal{O}(V_2)$ such that $\varphi(f_1(a)) \neq \varphi(f_2(a))$. Hence $f_1 \neq f_2$.

Surjectivity. Let $p_1, \ldots, p_m$ be projections of C^m defined by $p_\nu: C^m \ni x \to x_\nu \in C$. The restrictions of projections $p_1, \ldots, p_m$ to V_2 will be denoted by $\pi_1, \ldots, \pi_m$. Let $h \in \mathrm{Hom}_{C\text{-alg}}(\mathcal{O}(V_2), \mathcal{O}(V_1))$. Observe that $\pi_1, \ldots, \pi_m \in \mathcal{O}(V_2)$ and $h(\pi_1), \ldots, h(\pi_m) \in \mathcal{O}(V_1)$. Let $I(V_2)$ denote the ideal of all entire functions on C^n which vanish on V_2. If $g \in I(V_2)$, then $g | V_2 = 0 \in \mathcal{O}(V_2)$.
But $h(g | V_2) = 0 \in \mathcal{O}(V_1)$, so $h(g | V_2) \in I(V_1)$.

We define the map: $f\colon V_1 \ni x \to (h(\pi_1)(x), \dots, h(\pi_m)(x)) \in C_m$. We want to prove that $x \in V_1$ implies $f(x) \in V_2$. It suffices to show that for any $g \in I(V_2)$ and for any $x \in V_1$, $g(f(x)) = 0$. From $g \mid V_2 = 0$ we get that $h(g \mid V_2)(x) = 0$ for $x \in V_1$. It is enough to prove the equality

$$h(g \mid V_2)(x) = g(f(x)) \quad \text{for every } x \in V_1 .$$

The left hand side of this equality is of the form: $h(g(\pi_1, \dots, \pi_m))(x)$, where we substitute $\pi_1, \dots, \pi_m$ in the entire function g, the right hand side is of the form: $g(h(\pi_1)(x), \dots, h(\pi_m))(x)$, where $h(\pi_1)(x), \dots, h(\pi_m)(x)$ are complex numbers.

Let g_ν be the ν-th polynomial of the Taylor expansion of an entire function $g \in \mathcal{O}(C^n)$. Since g_ν tends to g in $\mathcal{O}(C^n)$ in the sense of the natural Frechet topology on $\mathcal{O}(C^n)$, so $g_\nu(\pi_1, \dots, \pi_m)$ tends to $g(\pi_1, \dots, \pi_m)$ in the natural Frechet topology on $\mathcal{O}(V_2)$. Because $h\colon \mathcal{O}(V_2) \to \mathcal{O}(V_1)$ is a homomorphism, then from the algebraic properties of homomorphisms we have $h(g_\nu(\pi_1, \dots, \pi_m)) = g_\nu(h(\pi_1), \dots, h(\pi_m))$. From our considerations we get that $g_\nu(h(\pi_1), \dots, h(\pi_m))$ tends to $g(h(\pi_1), \dots, h(\pi_m))$.

But the homomorphism h is continuous by Theorem (*). This implies that $h(g_\nu(\pi_1, \dots, \pi_m))$ tends to $h(h(\pi_1), \dots, h(\pi_m))$ and $h(g_\nu(\pi_1, \dots, \pi_m))$ tends to $h(g(\pi_1, \dots, \pi_m))$.

From the uniqueness of limits we have equality

$$g(h(\pi_1), \dots, h(\pi_m)) = h(g(\pi_1, \dots, \pi_m)) .$$

Now it is enough to prove that $f^* = h$.

Let $A(V)$ be the algebra of all polynomials restricted to V.

LEMMA 1. *$A(V)$ is dense in $\mathcal{O}(V)$.*

Proof. If $g \in \mathcal{O}(V)$ then there exists $\tilde{g} \in \mathcal{O}(C^n)$ such that $\tilde{g}$ is an extension of g on C^n. The sequence of Taylor polynomials of $\tilde{g}$ restricted to V approximates the element g uniformly on every compact set $K \subset V$.

LEMMA 2. *The homomorphisms f^* and h coincide on $A(V_2)$.*

Proof. Let P be a polynomial on C^m. Then if we substitute $\pi_1, \dots, \pi_m$ in P we have that $P(\pi_1, \dots, \pi_m) \in A(V_2)$. It suffices to prove the equality:

$$h(P(\pi_1, \dots, \pi_m)) = f^*(P(\pi_1, \dots, \pi_m)) .$$

Let $x \in V_1$, then $f^*(P(\pi_1, \dots, \pi_m))(x) = [P(\pi_1, \dots, \pi_m) \circ f](x) = [(P \mid V_2) \circ f](x) = P((f(x)) = P(h(\pi_1)(x), \dots, h(\pi_m)(x)) = P(h(\pi_1), \dots, h(\pi_m))(x) = h(P(\pi_1, \dots, \pi_m))(x)$.

The theorem is proved because f^* and h coincide on the dense subset $A(V_2)$ of $\mathcal{O}(V_2)$.

COROLLARY 1. *Let V be an analytic set in C^n, then the mapping*

$$V \ni x \to \hat{x} \in Hom_{C\text{-alg}}(\mathcal{O}(V), C)$$

is bijective ($\hat{x}$ — denotes evaluation map).

Proof. Let a be a point in C^m. The map $\mathrm{Hol}(\{a\}, V) \ni f \to f^* \in \mathrm{Hom}_{C\text{-alg}}(\mathcal{O}(V), C)$ is bijective, because a is an analytic subvariety of C^m. But it is evident that there exists a bijective map of V on $\mathrm{Hol}(\{a\}, V)$.

COROLLARY 2. (Theorem of Igusa [3]). *Let D be a pseudoconvex domain in C^k, then the mapping $D \ni x \to \hat{x} \in Hom_{C\text{-alg}}(\mathcal{O}(D), C)$, where $\hat{x}$ is an evaluation map, is bijective.*

Proof. From the well-known result of Remmert we know that every pseudoconvex domain D in C^k is embeddible in C^{2k+1} as an analytic set V.

It is evident that Corollary 1 gives a generalization of the Igusa theorem.

References

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[2] C. Baňică, O. Stănăsilă, *A result on section algebras over complex spaces*, Rend. Sci. Fis. Mat. Lincei, Vol. XLVII (1969), 233—235.
[3] J. Igusa, *On a Property of the Domain of Regularity*, Mem. Coll. of Science Un. Kyoto, Ser. A. Vol. XXVII, Math. No 2 (1952), 95—97.