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On a continuous extension of an algebraic function

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Abstract. Let A be a semi-analytic and relatively compact subset of R^n, $f\colon A\to R$ a continuous function. We assume that there exist an open set $G\supset\bar{A}$ and a polynom $P(x,t)=a_0(x)t^s+\ldots+a_s(x)$ with continuous coefficients defined in G, of non-zero degree in every point of G and such that $P(x,f(x))=0$ for $x\in A$.

In our paper we prove that for such a function f we can obtain a finite partition of A into semi-analytic sets A_i such that f_{A_i} admits a continuous extension on $\bar{A}_i$.

We give two applications of our theorem. From this theorem we obtain additional information on the form of a semi-algebraic set and on the rational zeros of a polynom $P(x,t)$.

Let A be a semi-analytic subset of R^n and let $f\colon A\ R$ be a continuous function. We shall discuss here a possibility of continuous extension of f on the closure $\bar{A}$ of the set A. Under suitable additional assumptions on the function f we shall give a construction of a partition $A_1,\ldots,A_k$ of A such that it is possible to obtain the extension of f_{A_i} on $\bar{A}_i$ (here we use the usual notation: f_B denotes the restriction of f to the subset B of the domain of the function f).

Observe that for some continuous functions there are no possibilities of such extensions. For instance, if

$$f\colon (0,1)\times(0,1)\ni(x,y)\to\frac{x^2}{x^2+y^2}\in R$$

is the well-known semi-algebraic function defined on the semi-algebraic set, it is impossible to extend f on $\bar{A}$ in a continuous manner, neither can we obtain the partition $A_1,\ldots,A_k$ mentioned above, because the set of cluster points of the function f in 0 is an interval.

THEOREM. *Let A be a semi-analytic and relatively compact subset of R^n, $f\colon A\to R$ a continuous function. We assume that there exist an open set $G\supset\bar{A}$ and a polynom $P(x,t)=a_0(x)t^s+\ldots+a_s(x)$ with coefficients continuous in G, of non-zero degree in every point of G and such that $P(x,f(x))=0$ for $x\in A$. Then there is a partition of the set A on semi-analytic sets $A_1,\ldots,A_k$ such that f_{A_i} admits a continuous extension on $\bar{A}_i$.*

Proof. Let $\Gamma_a f$ be the set of cluster points of f in a. Observe that

$$\bigcup_{a\in\bar{A}}\{a\}\times\Gamma_a f=\overline{\operatorname{graf} f}.$$

In our case the set $\Gamma_a f$ is finite for every $a \in \bar{A}$. It is easy to see that $P(a, b) = 0$ for $a \in \bar{A}$ and $b \in \Gamma_a f$. (For such points a, b there exists a sequence $\{x_k\} \subset A$, $x_k \to a$ such that $f(x_k) \to b$. Hence $P(a, b) = \lim P(x_k, f(x_k)) = 0$). Therefore we obtain the inclusion:

$$\Gamma_a f \subset \{b \in R; P(a, b) = 0\} .$$

Since $P(a, t)$ is the polynom of one variable of non-zero degree, the set of its zeros is finite. The set $\Gamma_a f$ is finite as well, being the subset of the set of zeros.

As the set A is semi-analytic, it admits a stratification compatible with A in every point $a \in \bar{A}$ (Łojasiewicz [1]). Since A is relatively compact, we may choose normal neighbourhoods $Q_1, \ldots, Q_k$ such that $A \subset Q_1 \cup \ldots \cup Q_k$. For every Q_i we take a prismatic stratification $\mathcal{N}^{(i)}$ ([1]). Let us recall the properties of such a stratification:

(*) For every $A \in H$ and for $a \in \bar{A} \setminus A$ there is a basis system of neighbourhoods $\{U_a\}$ such that $U_a \cap A$ are connected.

We know that $A = \bigcup\limits_{\text{finite}} A^i_j$; $A^i_j \in \mathcal{N}^{(i)}$, $A^i_j \cap A \neq \emptyset$.

Now we shall show that it is precisely the partition we need. Let A_0 be one of the sets A^i_j. Let us consider the restriction f_{A_0}. From (*) and from the well-known properties of connected sets and also from the equality $\Gamma_a f_{A_0} = \bigcap \overline{f(V_a \cap A_0)}$; $V_a \in$ basis of neighbourhoods it follows that $\Gamma_a f_{A_0}$ is connected.

Since $\Gamma_a f_{A_0} \subset \Gamma_a f$ and $\Gamma_a f$ is finite, $\Gamma_a f_{A_0}$ consists of one point (it is not empty because $\overline{\text{graf} f_{A_0}}$ is compact).

Let $\tilde{f}_{A_0}$ be the function defined as follows:

$$\tilde{f}_{A_0}(x) = \begin{cases} f(x) & \text{for } x \in A_0 , \\ \text{the unique point of } \Gamma_a f_{A_0} & \text{for } x \in \bar{A}_0 \setminus A_0 . \end{cases}$$

We shall prove that $\tilde{f}_{A_0}$ is continuous. Let us fix $a \in \bar{A}_0$ and a sequence $x_n \to a$. We may assume that $\{x_n\} \subset A_0$ or $\{x_n\} \subset \bar{A}_0 \setminus A_0$ (in the other case we consider subsequences). If $\{x_n\} \subset A_0$, we have $\tilde{f}_{A_0}(x_n) = f_{A_0}(x_n) \to f_{A_0}(a)$ because $\tilde{f}_{A_0}(a)$ is the unique cluster point of f_{A_0} in a. Let us now assume that $\{x_n\} \subset \bar{A}_0 \setminus A_0$. Let us suppose that we can choose a subsequence $\{\tilde{f}_{A_0}(x_{n_k})\} \subset \{\tilde{f}_{A_0}(x_n)\}$ which converges to a point $d \neq \tilde{f}_{A_0}(a)$. Then the sequence $\{(x_{n_k}, \tilde{f}_{A_0}(x_{n_k}))\}$ is contained in $\overline{\text{graf} f_{A_0}}$ and converges to (a, d). Hence $(a, d) \in \overline{\text{graf} f_{A_0}}$ and $d \in \Gamma_a f_{A_0}$, but it is impossible because $\Gamma_a f_{A_0} = \{\tilde{f}_{A_0}(a)\}$.

We have proved here that every convergent subsequence of the sequence $\{\tilde{f}_{A_0}(x^n)\}$ converges to $\tilde{f}_{A_0}(a)$. Therefore $\tilde{f}_{A_0}(x_n) \to \tilde{f}_{A_0}(a)$ (the sequence being contained in the compact set $\overline{\text{graf} f_{A_0}}$).

Remark. If the set A is semi-algebraic, the sets A^i_j will be semi-algebraic too. The partition A^i_j can be given effectively since the construction of the prismatic stratification is known ([1]).

Our Theorem does not of course guarantee the existence of an extension on all $\bar{A}$, even when the additional conditions are fulfilled.

APPLICATION I. Let us recall the Lemma on the form of semi-algebraic sets in R^{n+1} ([1]).

LEMMA. *Let E be a semi-algebraic set in R^{n+1}. Then there exists a partition of R^n into* *a family of semi-algebraic, disjoint and connected sets* $B_1, \ldots, B_k$ *and there exist the continuous functions*

$$h_j^{(i)}: B_i \to R \qquad h_1^{(i)} < \ldots < h_{k_i}^{(i)}$$

(*such that* E *is the union of some sets of the form*

$$\{x \in B_i;\ h_j^{(i)}(x) < t < h_{j+1}^{(i)}(x)\} \qquad j = 0, \ldots, k_i + 1$$
$$\{x \in B_i;\ t = h_j^{(i)}(x)\} \qquad j = 1, \ldots, k_i$$

we put $h_0 = -\infty$, $h_{k_i+1} = +\infty$).

Out Theorem permits the partition $B_1, \ldots, B_k$ to be obtained with an additional property such that the functions $h_j^{(i)}$ are continuous in $\bar{B}_i$. This gives us more information on the form of a semi-algebraic set.

APPLICATION II. We apply our Theorem to the theorem on zeros of a polynom ([1]), which we may write as follows:

Let $P(x, t)$ *be a polynom satisfying the assumptions of our Theorem. Let us assume that the set*

$A = \{x \in G;\ P(x, t)$ has exactly k different complex zeros$\}$ *is contained in* G *with its closure. We know that the set* A *is semi-algebraic and also that if* A *is connected, we can find continuous functions* $f_1, \ldots, f_r$ *such that for every* $x \in A$ $f_1(x), \ldots, f_r(x)$ *are all rational zeros of the polynom* $P(x, t)$.

It follows from our Theorem that A may be divided into a finite number of semi-algebraic sets such that functions f_i restricted to these sets can be extended in a continuous manner on the boundaries of their fields.

References

[1] S. Łojasiewicz, *Ensembles semi-analytiques*, IHES, Bures-sur-Yvette, 1965.