ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DII PRACE MATEMATYCZNE, Z. 20 1979

On a modification of the method of Euler polygons for systems of ordinary differential equations

by B. Śnieżek

The purpose of the present paper is to give simple generalizations of the results from [1].

1. We shall consider a system of ordinary differential equations

$$\frac{dx_i}{dt} = F_i(t, x_1, x_2, \ldots, x_m) \quad (i = 1, \ldots, m) \tag{1.1}$$

in an interval $[\alpha, \beta]$, with initial conditions

$$x_i(\alpha) = k_i \quad (i = 1, \ldots, m). \tag{1.2}$$

We construct Euler's polygons $u_n^i(t, \xi)$, $(i = 1, \ldots, m;\ n = 1, 2, \ldots)$ for the interval $[\alpha, \xi]$ $(\xi \in [\alpha, \beta])$ divided by points $\alpha_j^n = \alpha + \frac{j}{n}(\xi - \alpha)$ $(j = 0, 1, \ldots, n)$, as follows:

$$u_0^i(t, \xi) \overset{\text{df}}{=} k_i \qquad t \in [\alpha, \xi],$$

$$u_1^i(t, \xi) = k_i + F_i(\alpha, k_1, \ldots, k_m)(t - \alpha) \qquad t \in [\alpha, \xi],$$

$$u_2^i(t, \xi) = \begin{cases} k_i + F_i(\alpha, k_1, \ldots, k_m)(t - \alpha) & t \in \left[\alpha, \frac{\alpha+\xi}{2}\right], \\ k_i + F_i(\alpha, k_1, \ldots, k_m)\frac{\xi - \alpha}{2} + & \\ + F_i\left(\frac{\alpha+\xi}{2}, x_1\left(\frac{\alpha+\xi}{2}\right), \ldots, x_m\left(\frac{\alpha+\xi}{2}\right)\right)\left(t - \frac{\alpha+\xi}{2}\right) & t \in \left(\frac{\alpha+\xi}{2}, \xi\right], \end{cases}$$

. . . .

$$(1.3)\quad u_n^i(t,\xi)=\begin{cases} k_i+F_i(\alpha,k_1,\dots,k_m)(t-\alpha) & t\in[\alpha,\alpha_1^n]\\ k_i+F_i(\alpha,k_1,\dots,k_m)\dfrac{\xi-\alpha}{n}+ & \\ \quad +F_i(\alpha_1^n,x_1(\alpha_1^n),\dots,x_m(\alpha_1^n))(t-\alpha_1^n) & t\in(\alpha_1^n,\alpha_2^n]\\ \dots\dots\dots & \\ k_i+F_i(\alpha,k_1,\dots,k_m)\dfrac{\xi-\alpha}{n}+F_i(\alpha_1^n,x_1(\alpha_1^n),\dots,x_m(\alpha_1^n))\dfrac{\xi-\alpha}{n}+ & \\ \quad +F_i(\alpha_{n-1}^n,x_1(\alpha_{n-1}^n),\dots,x_m(\alpha_{n-1}^n))(t-\alpha_{n-1}^n) & t\in(\alpha_{n-1}^n,\xi] \end{cases}$$

where $i=1,\dots,m$, and

$$x_i(\alpha_1^n)\overset{\mathrm{df}}{=}u_1^i(\alpha_1^n,\xi),$$

$$x_i(\alpha_2^n)\overset{\mathrm{df}}{=}u_1^i(\alpha_1^n,\xi)+F_i(\alpha_1^n,u_1^1(\alpha_1^n,\xi),\dots,u_1^m(\alpha_1^n,\xi))\frac{\xi-\alpha}{n},$$

$$\dots\dots\dots$$

$$x_i(\alpha_{j+1}^n)=x_i(\alpha_j^n)+F_i(\alpha_j^n,x_1(\alpha_j^n),\dots,x_m(\alpha_j^n))\frac{\xi-\alpha}{n}.$$

Now we define

$$(1.4)\qquad \varphi_n^i(\xi)\overset{\mathrm{df}}{=}u_n^i(\xi,\xi)\quad (i=1,\dots,m).$$

Hence from (1.3) we have

$$\varphi_0^i(\xi)=k_i,$$

$$(1.5)\quad \varphi_n^i(\xi)=\varphi_{n-1}^i\left(\alpha+\frac{n-1}{n}(\xi-\alpha)\right)+\frac{\xi-\alpha}{n}F_i\times$$

$$\times\left[\alpha+\frac{n-1}{n}(\xi-\alpha),\ \varphi_{n-1}^1\left(\alpha+\frac{n-1}{n}(\xi-\alpha)\right),\dots,\varphi_{n-1}^m\left(\alpha+\frac{n-1}{n}(\xi-\alpha)\right)\right]$$

for $i=1,\dots,m$,

or equivalently

$$\varphi_0^i(\xi)=k_i,$$

$$\varphi_n^i(\xi)=k_i+\frac{\xi-\alpha}{n}\sum_{j=1}^{n-1}F_i\left[\alpha+\frac{j}{n}(\xi-\alpha),\ \varphi_j^1\left(\alpha+\frac{j}{n}(\xi-\alpha)\right),\dots,\varphi_j^m\left(\alpha+\frac{j}{n}(\xi-\alpha)\right)\right].$$

As in [1], we shall prove that the sequences $\{\varphi_n^i(t)\}$ $(i=1,\dots,m)$ defined by (1.4) are uniformly convergent to the solution of the system of equations (1.1) with the initial conditions (1.2), and the sequences of the derivatives $\left\{\frac{d}{dt}\varphi_n^i(t)\right\}$ $(i=1,\dots,m)$ are uniformly convergent to the corresponding derivatives of the solution of this problem (1.1)—(1.2).

THEOREM 1. *Let* M_i $(i = 1, ..., m)$ *be positive constants and let us put*

$$T = \{(t, x_1, ..., x_m): \alpha \leqslant t \leqslant \beta, |x_i - k_i| \leqslant M_i\} \quad (i = 1, ..., m).$$

Suppose that

1° F_i: $T \to R$ $(i = 1, ..., m)$ *are continuous,*

2° $|F_i(t, x_1, ..., x_m)| \leqslant M_i \quad (i = 1, ..., m), (\beta - \alpha) < 1,$

3° *in the interval* $[\alpha, \beta]$ *there exists exactly one solution* $x_i(\cdot)$ $(i = 1, ..., m)$ *of the problem* (1.1)—(1.2).

Then the sequences $\{\varphi_n^i(t)\}$ $(i = 1, ..., m)$ *defined by* (1.4) *are uniformly convergent to* $x_i(\cdot)$ $(i = 1, ..., m)$ *in* $[\alpha, \beta]$.

The proof of this theorem is similar to that of the first theorem in the paper [1] and so it will be omitted. Note only that the uniform convergence of the usual Euler polygons $\{u_n^i(t, \beta)\}$ $(i = 1, ..., m)$ to the unique solution of (1.1)—(1.2) can be obtained, under the assumptions of our theorem, by the classical methods (compare for instance [2], II, § 1).

2. Let

$$a_{ij} \overset{\text{df}}{=} \begin{cases} (\beta - \alpha) B_{ij} & \text{for} \quad i \neq j \\ -1 + (\beta - \alpha) B_{jj} & \text{for} \quad i = j, \end{cases} \tag{2.0}$$

$$c = (c_1, ..., c_m), \ c_i > 0, \text{ where } i = 1, ..., m, \ j = 1, ..., m.$$

Definition. We say that the matrix $[a_{ij}]$ satisfies the condition (V_c) if and only if there exists a solution $b = (b_1, ..., b_m)$ of the system of equations $[a_{ij}]b = -c$ such that $b_i > 0$ for $i = 1, ..., m$.

THEOREM 2. *If we suppose the assumptions* 1°, 2°, 3° *of Theorem* 1 *and if*

4° $F_i(t, x_1, ..., x_m)$ $(i = 1, ..., m)$ *have all* $m+1$ *first derivatives fulfilling the Lipschitz condition with respect to* $m+1$ *variables,*

5° $\left|\frac{\partial F_i}{\partial t}\right| \leqslant A_i$, $\left|\frac{\partial F_i}{\partial x_j}\right| \leqslant B_{ij}$ $(i = 1, ..., m, \ j = 1, ..., m)$,

6° $[a_{ij}]$ *defined by* (2.0) *fulfils the condition* (V_c) *for every* $c = (c_1, ..., c_m)$, $c_i > 0$ $(i = 1, ..., m)$,

then the sequences of derivatives $\left\{\frac{d}{dt}\varphi_n^i(t)\right\}$ $(i = 1, ..., m)$ *are uniformly convergent in* $[\alpha, \beta]$ *to the derivative of the solution of the problem* (1.1)—(1.2).

Note 1. On the assumption 4° of Theorem 2 we can immediately deduce the existence of the first derivatives of the functions $\varphi_n^i(t)$ $(i = 1, ..., m)$ defined by (1.4). (Proof by induction, making use of (1.5)).

Proof. In the proof of Theorem 2 we shall use Arzelo's theorem. We shall point out that the assumptions of this theorem are fulfilled.

Part I. First we shall show that the sequences $\left\{\frac{d}{dt}\varphi_n^i(t)\right\}$ $(i = 1, \ldots, m)$ are uniformly bounded in the interval $[\alpha, \beta]$, i.e. there exist positive constants Q^i $(i = 1, \ldots, m)$ such that

$$\left|\frac{d}{dt}\varphi_n^i(t)\right| \leqslant Q^i \quad \text{for } t \in [\alpha, \beta],\ n \in N,\ i = 1, \ldots, m. \tag{2.1}$$

We shall show that for any positive solution $Q^1, \ldots, Q^m$ of the system of equations

$$[a_{ij}]\, Q = -c \tag{2.2}$$

where $[a_{ij}]$ defined by (2.0) and $c_i = M_i + (\beta-\alpha)A_i$, $(i = 1, \ldots, m)$, the conditions (2.1) hold true; the existence of such a positive solution follows directly from assumption 6°.

The proof of proposition that Q^i defined above are such that (2.1) is satisfied proceeds by induction. In order to use the induction procedure we shall first show some inequalities between $\left|\frac{d}{dt}\varphi_n^i\right|$ and $\left|\frac{d}{dt}\varphi_{n-1}^i\right|$.

Let

$$\lambda_n(t) = \alpha + \frac{n-1}{n}(t-\alpha),\ \mu_n^i(t) = \varphi_{n-1}^i(\lambda_n(t)), \quad (i = 1, \ldots, m). \tag{2.3}$$

Thus

$$\lambda_n'(t) = \frac{n-1}{n},\ \frac{d}{dt}\mu_n^i(t) = \frac{n-1}{n}\frac{d}{dt}\varphi_{n-1}^i(\lambda_n(t)), \quad (i = 1, \ldots, m). \tag{2.4}$$

From (1.5) it follows that

$$\frac{d}{dt}\varphi_n^i(t) = \frac{d}{dt}\mu_n^i(t) + \frac{1}{n}F_i[\lambda_n(t), \mu_n^1(t), \ldots, \mu_n^m(t)] + \frac{t-\alpha}{n}\left\{\frac{\partial}{\partial t}F_i[\lambda_n(t), \mu_n^1(t), \ldots, \mu_n^m(t)]\lambda_n'(t) + \right.$$
$$\left. + \sum_{j=1}^{m}\frac{\partial}{\partial x_j}F_i[\lambda_n(t), \mu_n^1(t), \ldots, \mu_n^m(t)]\frac{d}{dt}\mu_n^j(t)\right., \tag{2.5}$$

$(i = 1, \ldots, m)$.

From (2.4), (2.5) and on the assumptions of Theorem 2 we have

$$\left|\frac{d}{dt}\varphi_n^i(t)\right| \leqslant \frac{n-1}{n}\left|\frac{d}{dt}\varphi_{n-1}^i(\lambda_n(t))\right| + \frac{1}{n}M_i + \frac{\beta-\alpha}{n}\frac{n-1}{n}A_i +$$
$$+ \frac{\beta-\alpha}{n}\frac{n-1}{n}\sum_{j=1}^{m}\left|\frac{d}{dt}\varphi_{n-1}^j(\lambda_n(t))\right| B_{ij}.$$

Since $\frac{n-1}{n} < 1$ we can state that

$$\left|\frac{d}{dt}\varphi_n^i(t)\right| \leqslant \frac{n-1}{n}\left|\frac{d}{dt}\varphi_{n-1}^i(\lambda_n(t))\right| + \frac{\beta-\alpha}{n}\sum_{j=1}^{m}\left|\frac{d}{dt}\varphi_{n-1}^j(\lambda_n(t))\right| B_{ij} + \frac{1}{n}M_i + \frac{\beta-\alpha}{n}A_i. \tag{2.6}$$

Now we can use the induction with respect to n. For $n = 1$ the inequalities required in (2.1) are obviously fulfilled. Assume that $\left|\frac{d}{dt}\varphi^i_{n-1}(t)\right| \leqslant Q^i$ $(i = 1, \ldots, m)$ in $[\alpha, \beta]$.

Observe that $Q^1, \ldots, Q^m$ satisfy the following system of inequalities

(2.7) $$\frac{n-1}{n}Q^i + \frac{\beta-\alpha}{n}\sum_{j=1}^{m} B_{ij}Q^j + \frac{M_i}{n} + \frac{\beta-\alpha}{n}A_i \leqslant Q^i \quad (i = 1, \ldots, m).$$

Thus, in virtue of (2.6) and the induction assumption of $\left|\frac{d}{dt}\varphi^i_{n-1}(t)\right|$, we obtain $\left|\frac{d}{dt}\varphi^i_n(t)\right| \leqslant Q^i$ $(i = 1, \ldots, m)$, $t \in [\alpha, \beta]$. The proof of (2.1) is completed.

Part II. Now we shall show that the functions $\frac{d}{dt}\varphi^i_n(t)$ $(i=1, \ldots, m)$ fulfil the Lipschitz condition with constants R^i $(i = 1, \ldots, m)$ common to all n; i.e. we shall show that there exist positive constants R^i $(i = 1, \ldots, m)$ such that

(2.8) $$\left|\frac{d}{dt}\varphi^i_n(t) - \frac{d}{dt}\varphi^i_n(\bar{t})\right| \leqslant R^i|t-\bar{t}| \quad \forall t, \bar{t} \in [\alpha, \beta] \quad (n = 0, 1, \ldots, i = 1, \ldots, m).$$

We shall prove that taking R^i $(i = 1, \ldots, m)$ as the solution of the following system of equations

(2.9) $$[a_{ij}]R = -c$$

where $[a_{ij}]$ defined by (2.0) and

$$c_i = 2\Big(A_i + \sum_{j=1}^{m} B_{ij}Q^j\Big) + (\beta-\alpha)\Big[L^i_{00} + \sum_{k=1}^{m} Q^k\Big(L^i_{1k} + L^i_{k1} + \sum_{j=1}^{m} L^i_{kj}Q^j\Big)\Big] \quad (i = 1, \ldots, m).$$

(The existence of such a positive solution follows directly from assumption 6°).

The proof is again by induction. Let $[L^i_{jk}]$ $(i = 1, \ldots, m, j = 0, 1, \ldots, m, k = 0, 1, \ldots, m)$ be the Lipschitz constants for the partial derivatives of the functions $F_i(t, x_1, \ldots, x_m)$ $(i = 1, \ldots, m)$. (See assumption 4°). Then we have:

$$\left|\frac{\partial}{\partial t}F_i(t, x_1, \ldots, x_m) - \frac{\partial}{\partial t}F_i(\bar{t}, \bar{x}_1, \ldots, \bar{x}_m)\right| \leqslant L^i_{00}|t-\bar{t}| + \sum_{j=1}^{m} L^i_{0j}|x_j - \bar{x}_j|$$

$$\left|\frac{\partial}{\partial x_k}F_i(t, x_1, \ldots, x_m) - \frac{\partial}{\partial x_k}F_i(\bar{t}, \bar{x}_1, \ldots, \bar{x}_m)\right| \leqslant L^i_{k0}|t-\bar{t}| + \sum_{j=1}^{m} L^i_{kj}|x_j - \bar{x}_j|$$

$$(i = 1, \ldots, m, \ k = 1, \ldots, m).$$

Under (2.3) and (2.4) we have:

(2.10) $$|\lambda_n(t) - \lambda_n(\bar{t})| \leqslant \frac{n-1}{n}|t-\bar{t}|,$$

$$(2.11)\qquad |\mu_n^i(t)-\mu_n^i(\bar{t})|\leqslant Q_n^i\frac{n-1}{n}|t-\bar{t}| \quad (i=1,\dots,m),$$

where Q_1^i $(i=1,\dots,m)$ are constants from part I of the proof.

Under (2.5), (2.10) and (2.11) we have

$$(2.12)\qquad \left|\frac{d}{dt}\varphi_n^i(t)-\frac{d}{dt}\varphi_n^i(\bar{t})\right|\leqslant\frac{n-1}{n}\left|\frac{d}{dt}\varphi_{n-1}^i(\lambda_n(t))-\frac{d}{dt}\varphi_{n-1}^i(\lambda_n(\bar{t}))\right|+$$
$$+\frac{n-1}{n}\frac{\beta-\alpha}{n}\sum_{j=1}^{m}B_{ij}\left|\frac{d}{dt}\varphi_{n-1}^j(\lambda_n(t))-\frac{d}{dt}\varphi_{n-1}^j(\lambda(\bar{t}))\right|+$$
$$+\frac{1}{n}\frac{n-1}{n}C_n^i|t-\bar{t}|, \quad (i=1,\dots,m),$$

where

$$(2.13)\qquad C_n^i=2\left(A_i+\sum_{j=1}^{m}B_{ij}Q_1^j\right)+\frac{n-1}{n}D^i \quad (i=1,\dots,m),$$

$$(2.14)\qquad D^i=(\beta-\alpha)[L_{00}^i+\sum_{k=1}^{m}Q_1^k(L_{1k}^i+L_{k1}^i+\sum_{j=1}^{m}L_{kj}^iQ_1^j)] \quad (i=1,\dots,m).$$

The implication

$$(2.15)\qquad \left\{\left|\frac{d}{dt}\varphi_{n-1}^i(t)-\frac{d}{dt}\varphi_{n-1}^i(\bar{t})\right|\leqslant R^i|t-\bar{t}| \quad in \quad [\alpha,\beta]\right.$$
$$\Rightarrow\left\{\left|\frac{d}{dt}\varphi_n^i(t)-\frac{d}{dt}\varphi_n^i(\bar{t})\right|\leqslant R^i|t-\bar{t}| \quad in \quad [\alpha,\beta],\right.$$

will be fulfilled by any positive solution R^i $(i=1,\dots,m)$ to the following system of inequalities:

$$(2.16)\qquad \frac{n-1}{n}R^i+\frac{\beta-\alpha}{n}\sum_{j=1}^{m}B_{ij}R^j+\frac{1}{n}\left[2\left(A_i+\sum_{j=1}^{m}B_{ij}Q_1^j\right)+D^i\right]\leqslant R^i \quad (i=1,\dots,m).$$

The inequalities (2.16) are fulfilled in particular by the constants R^i which are the solution of (2.9).

Hence in virtue of trivial inequality

$$\left|\frac{d}{dt}\varphi_0^i(t)-\frac{d}{dt}\varphi_0^i(\bar{t})\right|=0\leqslant R^i|t-\bar{t}|; \quad i=1,\dots,m;\ t,\bar{t}\in[\alpha,\beta]$$

we can use the induction procedure and, applying the implication (2.15) we finish the proof of our assertion.

Part III. In order to complete the proof of theorem we shall use methods similar to those in Part III_2 of the proof for Theorem 2 in [1].

3. THEOREM 3. *If we assume that* $F_i(t, x_1, \ldots, x_m)$ $(i = 1, \ldots, m)$ *have all bounded partial derivatives* $\frac{\partial^p}{\partial t^q \partial x_1^{r_1} \ldots \partial x_m^{r_m}} F_i$ $(p = 1, \ldots, k,\ q+r_1+\ldots+r_m = p,\ q = 0, 1, \ldots, k,\ r_1, \ldots, r_m = 0, 1, \ldots, k)$, *and all derivatives* $\frac{\partial^k}{\partial t^q \partial x_1^{r_1} \ldots \partial x_m^{r_m}} F_i$ $(i = 1, \ldots, m,\ p = 1, \ldots, k,\ q+r_1+\ldots+r_m = p,\ q, r_1, \ldots, r_m = 0, 1, \ldots, k)$ *fulfil the Lipschitz condition with respect to* $m+1$ *variables, then on the assumptions of Theorem* 2 *the sequences* $\left\{\frac{d^k}{dt^k} \varphi_n^i(t)\right\}$ $(i = 1, \ldots, m)$ *are uniformly convergent in* $[\alpha, \beta]$ *to* $\frac{d^k}{dt^k} x_i(t)$ $(i = 1, \ldots, m)$ *where* $x_i(t)$ $(i = 1, \ldots, m)$ *is the solution of the problem* (1.1)—(1.2).

Note 2. The existence of all derivatives $\frac{d^p}{dt^p} \varphi_n^i(t)$ $(p = 1, \ldots, k, i = 1, \ldots, m)$ on the assumption of the existence of all partial derivatives $\frac{\partial^p}{\partial t^q \partial x_1^{r_1} \ldots \partial x_m^{r_m}} F_i$ follows. (The induction proof with respect to n).

The idea of the proof of Theorem 3 is the same as in [1], also based on Arzelo's classical theorem.

4. Supplement: discussion of the condition (V_c).

Let $c_i > 0$ be given. Consider the system

$$[a_{ij}]b = -c \tag{4.1}$$

where $[a_{ij}]$ defined by (2.0) and $c = (c_1, \ldots, c_m)$, $c_i > 0$ $i = 1, \ldots, m$. Let

a) $$\sum_{\substack{p_\gamma = 1 \\ p_\gamma \neq i}}^{m-k+\eta} \ \sum_{\substack{p_{\gamma+1} > p \\ p_{\gamma+1} \neq i}}^{m-k+\eta+1} \ldots \sum_{\substack{p_\zeta > p_{\zeta-1} \\ p_\zeta \neq i}}^{m} \overset{\text{df}}{=} \sum\nolimits_{\gamma\zeta}^{\eta}$$

where $\eta \in \{1, 2, 3\}$; $\gamma \in \{1, 2\}$; $\zeta \in \{k, k-1\}$.

b) $$\sum_{\substack{p_2 = 1 \\ p_2 \neq i \\ p_2 \neq j}}^{m} \ \sum_{\substack{p_3 = 1 \\ p_3 \neq i \\ p_3 \neq j \\ p_3 \neq p_2}}^{m} \ldots \sum_{\substack{p_\zeta = 1 \\ p_\zeta \neq i \\ p_\zeta \neq j \\ p_\zeta \neq p_2 \\ \vdots \\ p_\zeta \neq p_{\zeta-1}}}^{m} \overset{\text{df}}{=} \sum\nolimits_{\zeta}^{1} \quad \text{where } \zeta \in \{k, k-1\}.$$

c) $$\begin{vmatrix} B_{\mu\nu} B_{\mu p_2} \ldots B_{\mu p_\zeta} \\ B_{p_2\nu} \\ \vdots \\ B_{p_\zeta\nu} \quad \ldots \quad B_{p_\zeta p_\zeta} \end{vmatrix} \overset{\text{df}}{=} \mathcal{M}_\zeta^{\mu\nu} \quad \text{where} \begin{cases} \zeta \in \{k, k-1\}, \\ \mu \in \{p_1, i\}, \\ \nu \in \{p_1, j, i\}. \end{cases}$$

The general formula for the solution of the system (4.1) is

$$b_i = \Big\{\sum_{\substack{j=1 \\ j \neq i}}^{m} \Big[c_j\Big(\sum_{k=1}^{m-1} (-1)^{k+1}(\beta-\alpha)^k \sum\nolimits_k \mathcal{M}_k^{ij}\Big)\Big] + c_i\Big(1 + \sum_{k=1}^{m-1} (-1)^k(\beta-\alpha)^k \sum\nolimits_{1k}^{1} \mathcal{M}_k^{p_1p_1}\Big)\Big\} \cdot W^{-1} \tag{4.2}$$

where

$$W \stackrel{\text{df}}{=} 1+\sum_{k=1}^{m}(-1)^k(\beta-\alpha)^k\Big(\sum_{p_1=1}^{m-k+1}\sum_{p_2>p_1}^{m-k+2}\dots\sum_{p_k>p_{k-1}}^{m}|B_{p_rp_s}|_{\substack{k\times k\\ r=1,\dots,k\\ s=1,\dots,k}}\Big)$$

$(i = 1, \dots, m)$.

The numbers b_i defined by (4.2) will be positive if we assume:

1° $(\beta-\alpha)\sum_{j=1}^{m} B_{jj}<1$

2° $(\beta-\alpha)\sum_{1k}^{1}\mathcal{M}_k^{p_1p_1}<\sum_{1k-1}^{2}\mathcal{M}_{k-1}^{p_1p_1}$

$k = 3, 5, 7, \dots, m-1$ for $m = 2n$, $n \in N$ or $k = 3, 5, 7, \dots, m-2$ for $m = 2n-1$, $n \in N$ and then $|B_{rs}]_{(m-1)\times(m-1)}>0$, $r = 1, \dots, m$, $r \neq i$; $s = 1, \dots, m$, $s \neq i$.

3° $(\beta-\alpha)\sum_{2k}^{2}\mathcal{M}_k^{ii}<\sum_{2k-1}^{3}\mathcal{M}_{k-1}^{ii}$

$k = 3, 5, 7, \dots, m$ for $m = 2n-1$, $n \in N$ or $k = 3, 5, 7, \dots, m-1$ for $m = 2n$, $n \in N$ and then $|B_{rs}|_{m\times m}>0$, $r = 1, \dots, m$; $s = 1, \dots, m$.

4° $(\beta-\alpha)\sum_{k}\mathcal{M}_k^{ij}<\sum_{k-1}\mathcal{M}_{k-1}^{ij}$

$k = 2, 4, 6, \dots, m-1$ for $m = 2n-1$, $n \in N$ or $k = 2, 4, 6, \dots, m-2$ for $m = 2n$, $n \in N$ and then $|^{B_{ij}}B_{rs}|_{(m-1)\times(m-1)}>0$, $r = 1, \dots, m$, $r \neq j$; $s = 1, \dots, m$, $s \neq i$.

Note 3. If in Theorem 2 we assume that

$$\left|\frac{\partial F_i}{\partial x_j}\right| \leqslant B_i \quad (i = 1, \dots, m,\ j = 1, \dots, m)\,, \tag{4.3}$$

then the system of equations (4.1) will be written in the form

$$[d_{ij}]b = -c, \text{ where } d_{ij} = \begin{cases} -1+(\beta-\alpha)B_i & \text{for } i = j \\ (\beta-\alpha)B_i & \text{for } i \neq j\,, \end{cases} \tag{4.4}$$

$c = (c_1, \dots, c_m)$, $c_i>0$, $i = 1, \dots, m$, $j = 1, \dots, m$.

Then $b_i>0$ $(i = 1, \dots, m)$ if we assume that $(\beta-\alpha)\sum_{j=1}^{m} B_j<1$. In the case of assumption (4.3) the general formula for b_i $(i = 1, \dots, m)$ is consecutive:

$$b_i = \Big\{\Big[1-(\beta-\alpha)\sum_{\substack{j=1\\ j\neq i}}^{m} B_j\Big]c_i+(\beta-\alpha)B_i\sum_{\substack{j=1\\ j\neq i}}^{m} c_j\Big\}\cdot\Big[1-(\beta-\alpha)\sum_{j=1}^{m} B_j\Big]^{-1}.$$

Note 4. It is clear that in the proof of Theorem 2 we have used the condition (V_c) only for $c = (c_1, \dots, c_m)$ such that

$c_i = M_i+(\beta-\alpha)A_i$ and

$$c_i = 2\Big(A_i+\sum_{j=1}^{m} B_{ij}Q^j\Big)+(\beta-\alpha)\Big[L_{00}^i+\sum_{k=1}^{m} Q^k\Big(L_{1k}^i+L_{k1}^i+\sum_{j=1}^{m} L_{kj}^iQ^j\Big)\Big] \quad (i = 1, \dots, m)\,.$$

(see Part II of the proof Theorem 2). Hence, formally, we can modify Theorem 2 assuming (V_c) only for c as above, not necessarily for every c.

References

[1] A. Pelczar, *On a modification of the method of Euler polygons for the ordinary differential equation,* Annales Polonici Mathematici XV (1964), 195—202.
[2] W. W. Stiepanow, *Równania różniczkowe*, PWN, Warszawa 1964.