On a theorem of S. N. Bernstein in F-spaces

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Introduction. Let $C_R([0, 1])$ denote the Banach space of all real-valued continuous functions on the interval $[0, 1]$ with the supremum norm $\|\cdot\|$. For a function $f \in C_R([0, 1])$, put

$$d_k(f) = \inf\{\|f-p\|;\ p \text{ is a polynomial of degree} \leqslant k\}.$$

By a theorem of S. N. Bernstein ([2], p. 292), for every non-increasing null-sequence $\{\varepsilon_k\}$ of non-negative numbers there is a function $f \in C_R([0, 1])$ with $d_k(f) = \varepsilon_k$ for all k. This problem of the existence of an element $f \in C_R([0, 1])$ with prescribed values $d_k(f)$, called by Bernstein the *inverse problem of the theory of best approximation*, was reformulated by V. N. Nikol'skiĭ [4] in the following way. A Banach space E is said to have the *property* (B) *with respect to a family* $\mathscr{F}$ of closed linear subspaces of E, if for every increasing sequence

$$V_1 \subset V_2 \subset \ldots$$

of distinct subspaces of E belonging to the family $\mathscr{F}$ and every sequence of non-negative numbers

$$\varepsilon_1 \geqslant \varepsilon_2 \geqslant \ldots \geqslant 0 = \lim_{k\to\infty} \varepsilon_k$$

there exists an element $x \in E$ such that

$$\operatorname{dist}(x, V_k) = \varepsilon_k \quad (k = 1, 2, \ldots).$$

If these equalities hold for almost all k, then we shall say that E has *almost* the *property* (B) *with respect to* $\mathscr{F}$. If the family $\mathscr{F}$ consists of all closed linear subspaces of E, then E is said to have the *absolute property* (B). Various examples of Banach spaces having the property (B) with respect to certain families $\mathscr{F}$ were given by V. N. Nikol'skiĭ and I. S. Tjuremskih. In particular Nikol'skiĭ [3] proved that if E has the absolute property (B), then it is reflexive. Tjuremskih [10] showed that each Hilbert space E has the absolute property (B). It is not so far known whether there exist other Banach spaces having the absolute property (B). (For more complete information see [8].) It is well-known, however, that every real Banach space E has the property (B) with respect to the family $\mathscr{F}_0$ of all finite-dimensional linear subspaces of E (see [8], theorem 5.8, p. 264, [9], theorem 2.5, p. 50). Actually the last assertion is true for all (real or complex) Banach spaces (see [5], theorem 5.1).

In this note we shall consider Bernstein's inverse problem of the theory of best approximation in the case where E is an F-*space*, i.e. a complete metric linear space, and show that E has the property (B) (with some natural restrictions on the sequence $\{\varepsilon_k\}$) with respect to certain families of finite-dimensional subspaces of E (theorem 1.3). In particular, we shall prove (corollary 1.5) that all complete p-normed spaces ($0<p\leqslant 1$) have the property (B) with respect to the family $\mathscr{F}_0$, and all locally pseudoconvex spaces have almost this property with respect to a subfamily of the family $\mathscr{F}_0$ (corollary 2.3). Finally we shall show that a large class of F-spaces (not necessarily locally pseudoconvex) formed by integrable functions on compact subsets of $\mathbf{C}^n$ has almost the property (B) with respect to the family of finite-dimensional subspaces of polynomials (corollary 3.6).

We hope that our results for F-spaces are new. We note, however, a paper by Albinus [1], in which some other questions related to generalizations of Bernstein's theorem to the case of F-spaces have been treated.

1. The main theorem

1.1. Let E be an F*-*space* (i.e. a metric linear space over the field K of real or complex numbers; see e.g. [6]). We shall denote by $|\cdot|$ an F-*norm* in E ($|x| = 0$ iff $x = 0$, $|\alpha x| = |x|$ for all α in K with $|\alpha| = 1$, $|x+y|\leqslant|x|+|y|$, and $|\alpha_n x|\to 0$ provided $\alpha_n\to 0$) giving the topology of E. If E is complete, it is called an F-*space*.

Given a closed linear subspace V of E and an element $x\in E$, we put

(1) $$d(x, V) = \inf\{|x-v|:\ v\in V\}.$$

We note that

(2) $$d(x+y, V)\leqslant d(x, V)+d(y, V),\quad x, y\in V,$$

and

(3) $$d(x+v, V) = d(x, V),\quad x\in E,\ v\in V.$$

We also have

$$|d(x, V)-d(y, V)|\leqslant|x-y|,\quad x, y\in E,$$

which yields the continuity of the mapping

(4) $$E\ni x\to d(x, V)\in R_+ .$$

We shall need the following.

1.2. Lemma (cf. e.g. [8], lemma 6.1, p. 151, the case where E is normed). *Let V be a finite-dimensional subspace of an* F-*space E. Suppose that for a certain $\varepsilon_0>0$ the set $B(\varepsilon_0)\cap V$ is compact. Then for each $x\in B(\frac{1}{2}\varepsilon_0)$ there is an element $v\in V$ such that*

$$d(x, V) = |x-v|\,;$$

throughout the paper $B(r)$ denotes the ball $\{x\in E:\ |x|\leqslant r\}$.

Proof. If $x \in E$, $v \in V$ and $|v| > 2|x|$, then by (1) we have

$$|x-v| \geqslant |v| - |x| > |x| \geqslant d(x, V),$$

whence

$$d(x, V) = \inf\{|x-v|: v \in B(2|x|) \cap V\}.$$

Therefore, if $|x| \leqslant \frac{1}{2}\varepsilon_0$, then the set $B(2|x|) \cap V$ is compact, whence by the continuity of the function (4) we get the result.

1.3. THEOREM. *Let $V_1 \subset V_2 \subset \ldots$ be an increasing sequence of finite-dimensional subspaces of an* F-*space E. Assume that*

(i) *there is an $\varepsilon_0 > 0$ such that the sets $B(\varepsilon_0) \cap V_k$ are compact ($k = 1, 2, \ldots$).*

Put for $k = 1, 2, \ldots,$

$$\eta_k = \sup\{d(x, V_k): x \in V_{k+1}\}.$$

Then for every sequence $\{\varepsilon_k\}$ of non-negative numbers such that

(ii) $$\tfrac{1}{2}\varepsilon_0 > \varepsilon_1 \geqslant \varepsilon_2 \geqslant \ldots \geqslant 0 = \lim_{k\to\infty} \varepsilon_k$$

and

(iii) $$2\varepsilon_k \leqslant \eta_k, \quad k = 1, 2, \ldots,$$

there exists an element $x \in E$ such that

$$d(x, V_k) = \varepsilon_k \quad \text{for } k = 1, 2, \ldots$$

Proof. For $k = 1, 2, \ldots$, put

$U_k = \{x \in E: d(x, V_k) \leqslant \varepsilon_k\}$,

$W_k = \{x \in E: d(x, V_k) > \varepsilon_k\}$,

$A_k = \{x \in E: d(x, V_k) = \varepsilon_k\}$

and

$B_k = A_1 \cap \ldots \cap A_k$.

We note that

(5) $$A_k \neq \emptyset \quad \text{for } k = 1, 2, \ldots$$

Indeed, if $\varepsilon_k = 0$, then $0 \in A_k$. Suppose $\varepsilon_k > 0$. If $A_k = \emptyset$, then $E \subset U_k \cup W_k$. Therefore, since the space E is connected, and since U_k and W_k are disjoint open sets, we should have $E \subset U_k$ because $0 \in E \cap U_k \neq \emptyset$. On the other hand, by (iii)

$$\sup\{d(x, V_k): x \in E\} \geqslant \eta_k \geqslant 2\varepsilon_k > \varepsilon_k,$$

whence $E \cap W_k \neq \emptyset$, a contradiction.

Next we observe that

(6) For each k, if $a \in A_k$, then there is an element $u \in V_k$ such that $a_k = a - u \in A_k$ and $|a_k| = \varepsilon_k$.

Since $\varepsilon = \frac{1}{2}\varepsilon_0 - \varepsilon_k > 0$, there is an element $w \in V_k$ such that

$$|a-w| < d(a, V_k) + \varepsilon = \varepsilon_k + \varepsilon = \tfrac{1}{2}\varepsilon_0.$$