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DII PRACE MATEMATYCZNE, Z. 20 1979

A decomposition of operator representations of the algebra $R(K_1 \times K_2)$

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Let X be a compact Hausdorff space and let $C(X)$ be the Banach algebra of all complex continuous functions on X with the norm $\|u\| = \sup_X |u|$ $(u \in C(X))$. A *function algebra* on X is a uniformly closed subalgebra of $C(X)$ containing constants and separating the points of X. $R(K)$ denotes the uniform closure of algebra of all rational functions on a given compact set $K \subset C^n$ with singularities off K. Throughout the paper K_1, K_2 will stand for the compact subsets of the complex plane.

Let T be a contractive representation of function algebra A in a Hilbert space. The decomposition of spectrum A into the sum of Gleason's parts induces the unique decomposition of T into an orthogonal sum of contractive representations [5]. The Gleason's parts of the algebra $R(K)$ for $K \subset C$ are well known (see [3. VI. 3]) and so are some properties of decomposition of the representation of $R(K)$ (see [4]). In the case $A = R(K_1 \times K_2)$ we may apply Bekken's decomposition of orthogonal measures [1]. Using the technique of [7] we decompose T on an orthogonal sum of four contractive representations and examine its spectral properties.

1. $M(X)$ stands for the set of all complex regular Baire measures on X. The only measures we shall consider in the present paper are those in $M(X)$.

A measure $\mu \in M(X)$ is said to be orthogonal to A if $\int_X u d\mu = 0$ for all $u \in A$. The set of all measures orthogonal to A is denoted by $A^\perp$. A positive measure λ on X is called a *representing measure* for $x \in X$, if $\int_X u d\lambda = u(x)$ for all $u \in A$. μ_E denotes the restriction of measure μ to a Baire subset $E \subset X$. Call $E \subset X$ a *nullset* for $A^\perp$ if $\mu_E = 0$ for all measures $\mu \in A^\perp$.

Let N be an arbitrary subset of $M(X)$. We say that the measure μ is singular to N, if it is *singular* to every measure $\nu \in N$. We say that two sets of measures N_1, N_2 are *mutually singular*, if every two measures $\mu_1 \in N_1$, $\mu_2 \in N_2$ are mutually singular. By $|\mu|$ we denote the variation measure of $\mu \in M(X)$ and by $\|\mu\|$ the total variation of μ over X.

A point $x \in X$ is called a *peak point* of a function algebra $A \subset C(X)$, if there exists a function $u \in A$ such that $u(x) = 1$ and $|u(z)| < 1$ for all $z \in X$, $z \neq x$.

Following [2] we use the notion of a band of measures. The set B of measures on a compact set X is called a *band*, if the following conditions are satisfied

(i) If $\mu \in B$ and a measure ν is absolutely continuous with respect to $|\mu|$ then $\nu \in B$.

(ii) If $\mu_n \in B$ for $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$, then $\sum_{n=1}^{\infty} \mu_n \in B$.

The following lemma shows that if B is a band, then every measure $\mu \in M(X)$ has a Lebesque decomposition with respect to B.

LEMMA 1.1. *Suppose that B is a band of measures on X. Then every measure $\mu \in M(X)$ has a unique decomposition*

$$\mu = \mu_B + \mu_s, \tag{1.1}$$

where $\mu_B \in B$ and μ_s is singular to B. Moreover we have

$$\|\mu\| = \|\mu_B\| + \|\mu_s\|. \tag{1.2}$$

The measure μ_B will be called the *absolutely continuous part* of μ with respect to the band B.

The proof of this lemma is a minor modification of the proof of [3, II, 7, 5].

COROLLARY 1.1. *Let B be a band in $M(X)$. Then we may define the linear projection P_B on B in the following way*

$$P_B \mu = \mu_B, \quad \mu \in M(X),$$

where μ_B is the absolutely continuous part of μ with respect to the band B. This projection is bounded, because (1.2) *implies the inequality*

$$\|P_B \mu\| \leqslant \|\mu\|, \quad \mu \in M(X). \tag{1.3}$$

Let N be a subset of $M(X)$. Denote by N^s set of all measures belonging to $M(X)$ which are singular to N. It is easy to check that the set N^S is a band of measures.

From lemma 1.1 we deduce the following

COROLLARY 1.2. *If B is a band of measures on the set X, then $M(X)$ is a unique direct sum*

$$M(X) = B + B^S. \text{ Moreover } (B^S)^S = B.$$

Corollary 1.2 implies now

COROLLARY 1.3. *Suppose N is an arbitrary subset of $M(X)$. Then $(N^S)^S$ is the smallest band including N. We call it the band generated by N and denote it by $\langle N \rangle$.*

2. Let K_1, K_2 be the compact subsets of the complex plane. Q_1, Q_2 will stand for the sets of all non-peak points of algebras $R(K_1)$, $R(K_2)$ respectively.

In the dual of $C(K_1 \times K_2)$ which can be identified with the space $M(K_1 \times K_2)$, following [1] we introduce three bands of measures

$\mathfrak{M}_0$ — band generated by representing measures (with respect to the subalgebra $R(K_1 \times K_2)$) for points in $Q_1 \times Q_2$.

$\mathfrak{M}_1$ — band of measures, which are supported on sets of the form $E_1 \times K_2$, where E_1 is a nullset for $R(K_1)^\perp$.

$\mathfrak{M}_2$ — band of measures, which are supported on sets of the form $K_1 \times E_2$, where E_2 is a nullset for $R(K_2)^\perp$.

Bekken proved in [1. Th. 1.] some properties of these bands. We quote his result as a lemma.

LEMMA 2.1. *If μ is a measure on $K_1 \times K_2$ orthogonal to $R(K_1 \times K_2)$, then μ has the unique decomposition*

$$\mu = \mu_0 + \mu_1 + \mu_2 ,$$

where the measures μ_j ($j = 0, 1, 2$) are pairwise mutually singular, orthogonal to the algebra $R(K_1 \times K_2)$ and for $j = 0, 1, 2$ $\mu_j \in \mathfrak{M}_j$ ($\mathfrak{M}_j$ are defined above).

Now we give some interpretation of lemma 2.1. Put

$$\mathfrak{N}_i = \mathfrak{M}_i \cap \langle R(K_1 \times K_2)^\perp \rangle \quad (i = 0, 1, 2) .$$

Then

(2.1) *The bands $\mathfrak{N}_i (i = 0, 1, 2)$ are pairwise mutually singular.*

$$\mathfrak{N}_0 + \mathfrak{N}_1 + \mathfrak{N}_2 = \langle R(K_1 \times K_2)^\perp \rangle . \tag{2.2}$$

Corollary 1.1 implies that for every $\mathfrak{N}_i$ ($i = 0, 1, 2$) there exists a uniquely determined linear bounded projection on band $\mathfrak{N}_i$. We denote it by P_i ($i = 0, 1, 2$).

It follows from (2.1) that

$$P_i P_j = 0 \quad \text{for} \quad i, j = 0, 1, 2 , \quad i \neq j . \tag{2.3}$$

We say that a bounded projection P in the dual of $C(X)$ has the property R (see [7]) with respect to the closed subalgebra $A \subset C(X)$ if

$$\mu \in A^\perp \Rightarrow P\mu \in A^\perp \tag{2.4}$$

$$\forall v \in C(X) , \quad \forall \mu \in M(X) \quad P(v d\mu) = v d(P\mu) . \tag{2.5}$$

It follows from the properties of bands that every projection on a band satisfies (2.5). Lemma 2.1 implies that the projections on bands $\mathfrak{N}_i$ ($i = 0, 1, 2$) satisfy (2.4) with respect to the algebra $R(K_1 \times K_2)$. So we obtain the following conclusion

LEMMA 2.2. *The projections P_i ($i = 0, 1, 2$) have the property R with respect to the subalgebra $R(K_1 \times K_2) \subset C(K_1 \times K_2)$.*

3. Let H be a complex Hilbert space with a scalar product (f, g) ($f, g \in H$) and a norm $\|f\| = \sqrt{(f,f)}$. By $L(H)$ we denote the algebra of all linear operators in H. $\|T\|$ stands for the bounded norm and T^* for the adjoint of operator $T \in L(H)$. I is the identity operator in H.

The algebra homomorphism T: $A \to L(H)$ is called a (*contrative*) *representation* of A if

$$T(1) = I , \tag{3.1}$$

$$\|T(u)\| \leqslant \|u\| = \sup_X |u| \quad \text{for} \quad u \in A . \tag{3.2}$$

It is well known (see for example [5]) that for $f, g \in H$ there are Baire regular measures $\mu_{f,g}$ on X called elementary measures of T so that

(3.3) $$\|\mu_{f,g}\| \leqslant \|f\| \; \|g\| ,$$

(3.4) $$(T(u)f, g) = \int_X u d\mu_{f,g} \quad \text{for} \quad u \in A .$$

Now we shall cite some results of [7] in a form useful for our purposes.

If P is a linear bounded projection in the dual of $C(X)$ and P has the property R with respect to a closed subalgebra A, then for each $u \in A$ there exists a unique operator $T_p(u) \in L(H)$ such that every elementary measure $\mu_{f,g}$ of T satisfies the condition

(3.5) $$(T_p(u)f, g) = \int_X u dP\mu_{f,g} \quad (f, g \in H) .$$

The mapping T_p: $A \ni u \to T_p(u) \in L(H)$ is linear, multiplicative and bounded.

So we conclude that $T_p(1)$ is an orthogonal projection in $L(H)$. Then the mapping T_p is a representation of algebra A in $L(H_p)$ where $H_p = T_p(H)$. We call T_p the *P–part* of the representation T.

We say that the representation T is *P–supported*, if T has a system of elementary measures $\{\mu_{f,g}\}$ $(f, g \in H)$ such that $\mu_{f,g} = P\mu_{f,g}$ for every $f, g \in H$.

(3.6) *T is P–supported if and only if $T = T_p$.*

Let J denote the identity mapping in the dual of $C(X)$. If P is a bounded projection, then $J-P$ is also a bounded projection. If P has the property R, then $J-P$ has also the property R and the following conditions are satisfied:

(3.7) *Representation $T-T_p$ is a $J-P$–part of T.*

(3.8) *Representation T is the unique direct sum of its P–part and $J-P$–part.*

A more detailed description of the decomposition of function algebras is included in theorem 5.3 of [7]. We give it here as

LEMMA 3.1. *Suppose $\{P_\alpha\}$ is an indexed set of bounded projections in the dual of $C(X)$ which have the property R with respect to the function algebra $A \subset C(X)$ and satisfy the condition*

$$P_\alpha P_\beta = 0 \quad \textit{for } \alpha \neq \beta .$$

Let T be a representation of algebra A in $L(H)$. (H is a Hilbert space). Then T is a unique orthogonal sum

$$T = \bigoplus_\alpha T_\alpha \oplus T' ,$$

where T is the P–part of T and T' is some representation.

Now we return to the algebra $R(K_1 \times K_2)$. The projections P_i $(i = 0, 1, 2)$ are bounded and have the property R with respect to $R(K_1 \times K_2)$ (lemma 2.2). Then we conclude from (2.3) that P_i $(i = 0, 1, 2)$ satisfy the assumption of lemma 3.1. Put $P_3 = J - \sum_{i=0}^{2} P_i$. It is easy to check that P_3 is also a bounded projection having the property R. So we get the following

THEOREM 1. *Suppose* $T: R(K_1 \times K_2) \to L(H)$ *is a representation of the algebra* $R(K_1 \times K_2)$. *Then*

1° *T is a unique orthogonal sum of representations of* $R(K_1 \times K_2)$

$$T = T_0 \oplus T_1 \oplus T_2 \oplus T_3 \tag{3.9}$$

2° a) *For* $i = 0, 1, 2$ *the representation* T_i *is the* P_i*–part of T.*

b) T_i $(i = 0, 1, 2)$ *has a system of elementary measures*

$$\{\mu^i_{f,g}\}_{f,g \in H} \text{ such that } \mu^i_{f,g} \in \mathfrak{N}_i .$$

3° a) *The representation* T_3 *is the* P_3*–part of T.*

b) T_3 *has a system of elementary measures which is singular to the band generated by* $R(K_1 \times K_2)^{\perp}$.

Remark. For $i = 0, ..., 3$ the operator $T_i(1)$ is an orthogonal projection in H. So we conclude that the space H decomposes uniquely to

$$H = H_0 \oplus H_1 \oplus H_2 \oplus H_3$$

where $H_i = P_i(1)H$ for $i = 0, ..., 3$. The values of representations T_i are bounded linear operators in H_i.

Proof of the theorem 1. Lemma 3.1 implies the existence of decomposition (3.9) and the condition 2° a.

Condition 3° a is a consequence of (3.7).

Conditions 2° b and 3° b we deduce from the definitions of projections P_i $(i = 0, ..., 3)$ and from (3.6).

(3.8) implies that the decomposition (3.9) is unique.

4. We say that T is an X–representation, if there is a representation $\tilde{T}: C(X) \to L(H)$ such that $\tilde{T}(u) = T(u)$ for $u \in A$.

Let $B(X)$ denote the totality of all Baire subsets of X.

The mapping $F: B(X) \to L(H)$ is called the spectral measure on X, if

(j) Its values are orthogonal projections in H.

(jj) $F(X) = I$.

(jjj) For arbitrary $f, g \in H$ the mapping $B(X) \ni \sigma \to (F(\sigma)f, g)$ is a regular Baire measure on X.

We say that the representation T has a spectral measure F, if for every $u \in A$ we have the following equality

$$T(u) = \int u dF \tag{4.1}$$

Now we give a well-known property (see [6] Th. 4.2 and 5.3) of representations of function algebras.

LEMMA 4.1. *A representation T of A has a unique spectral measure if and only if it is an X–representation.*

As a trivial consequence of the theorem 4.2 of [7] we have

Lemma 4.2. *Suppose that P is a bounded projection in the dual of $C(X)$ and P satisfies the conditions* (2.5) *and*

$$\mu \in A^{\perp} \Rightarrow P\mu = 0 . \tag{4.2}$$

If $T: A \to L(H)$ is a P-supported representation, then there exists a unique P-supported representation $\tilde{T}: C(X) \to L(H)$ such that $\tilde{T}(u) = T(u)$ for $u \in A$.

Let T be a representation of algebra $R(K_1 \times K_2)$. It follows from the definition of projection P_3 and from lemma 2.1, that P_3 satisfies the conditions (2.5) and (4.2).

Let T_3 be as in theorem 1. Since T_3 is the P_3-part of T, then T_3 and P_3 satisfy the assumption of lemma 4.2. So we may conclude that T_3 is an X-representation for $X = K_1 \times K_2$.

Hence applying theorem 1 and lemmas 4.1, 4.2 we obtain

Theorem 2. *Suppose $T: R(K_1 \times K_2) \to L(H)$ is a representation of algebra $R(K_1 \times K_2)$. Then*

1° *T is a unique orthogonal sum of representations of $R(K_1 \times K_2)$*

$$T = T_0 \oplus T_1 \oplus T_2 \oplus T_3$$

2° *For $i = 0, 1, 2$ T_i has a system of elementary measures belonging to the band $\mathfrak{N}_i$.*

3° *T_3 is an X-representation for $X = K_1 \times K_2$.*

The spectral measure of T_3 is singular to $\langle R(K_1 \times K_2)^{\perp} \rangle$.

References

[1] O. B. Bekken, *Rational approximation on product sets*, Trans. Amer. Math. Soc. 191 (1974), 301—316.

[2] E. Briem, A. M. Davie, B. K. Oksendal, *A functional calculus for pairs of commuting contractions*, J. London Math. Soc. 7 (1974), 709—718.

[3] T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliff, New Jersey, 1969.

[4] W. Mlak, *Absolutely continuous operator-valued representations of function algebras*, Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys. 17 (1969), 547—550.

[5] —, *Decompositions and extensions of operator-valued representations of function algebras*, Acta Sci. Math. (Szeged), 30 (1969), 181—193.

[6] —, *Introduction to the Hilbert space theory* (in Polish), Warszawa 1970.

[7] B. H. Szafraniec, *Decompositions of non-contractive operator-valued representations of Banach algebras*, Studia Math. 42 (1972), 97—108.

[8] —, *Some spectral properties of operator-valued representation algebras*, Ann. Polon. Math., 25 (1971), 187—194.