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DII PRACE MATEMATYCZNE, Z. 20 1979

Zeros of entire functions

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1. Introduction. We shall denote by $\mathbf{C}[z_1, \ldots, z_N]$ the ring of all polynomials of N complex variables $z_1, \ldots, z_N$ and by $H(\Omega)$ the ring of all holomorphic functions in a domain $\Omega \subset \mathbf{C}^N$. Given an ideal $\mathcal{U}$ in the polynomials ring $\mathbf{C}[z_1, \ldots, z_N]$, we recall that the algebraic variety of $\mathcal{U}$ in $\mathbf{C}^N$ is the set $V = V(\mathcal{U})$ of all points $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$ such that $p(z) = 0$ for all $p \in \mathcal{U}$. For every subset E of $\mathbf{C}^N$ we denote by $\mathcal{U}(E)$ the set of all polynomials in $\mathbf{C}[z_1, \ldots, z_N]$ which vanish at every point z of E. Clearly, $\mathcal{U}(E)$ is an ideal in $\mathbf{C}[z_1, \ldots, z_N]$. If $p_0, \ldots, p_k$ are elements of $\mathbf{C}[z_1, \ldots, z_N]$ then the smallest ideal which contains all the p_j will be written in the form $\mathcal{U}(p_0, \ldots, p_k)$. Analogously we shall denote by $I, I(E), I(f_1, \ldots, f_k)$ ideals in $H(\mathbf{C}^N)$.

Since the ring $\mathbf{C}[z_1, \ldots, z_N]$ is Noetherian it follows that every ideal has a finite basis. In $H(\mathbf{C}^N)$ there are ideals which are not finitely generated.

We have the following

THEOREM A (W. Rudin [4]). *If Ω is a domain of holomorphy in $\mathbf{C}^N$ then every finitely generated ideal in $H(\Omega)$ is closed relative to the topology of uniform convergence on compact sets.*

THEOREM B ([4], proof of Theorem 5). *Suppose V is an algebraic variety in $\mathbf{C}^N$. Then there are polynomials $p_0, \ldots, p_k \in \mathcal{U}(V)$ which form a basis of $I(V)$.*

2. Order of the zeros of an entire function. Let f be an entire function in $\mathbf{C}^N$. Denote by $Z(f)$ the set of all points $z \in \mathbf{C}^N$ at which $f(z) = 0$ and suppose $\mathring{z} \in Z(f)$.

Assume that $w \in C^N - \{0\}$ and define

$$\chi_w \colon \mathbf{C} \in \lambda \to \mathring{z} + \lambda w \in \mathbf{C}^N .$$

Associate to each point $w \in C^N - \{0\}$ the entire function $f \circ \chi_w$ in one complex variable. Function $f \circ \chi_w$ is identically equal to zero or it has an isolated zero point set.

We call $k(\mathring{z}, w)$ the order of the zero of f in the direction $[w]$ if the function $f \circ \chi_w$ has a zero of order $k(\mathring{z}, w)$ at 0. If $f \circ \chi_w = 0$ then by definition $k(\mathring{z}, w) = 0$.

THEOREM 1. *For every entire function f*

$$\sup\{k(\mathring{z}, w) \colon w \in \mathbf{C}^N - \{0\}\} = \varkappa(\mathring{z}) < \infty .$$

Proof. Without loss of generality we may assume that $\mathring{z} = 0$. Put $k(w) = k(0, w)$ and write the function f in the form

$$f = f_s + f_{s+1} + \dots, \tag{1}$$

where f_j are homogeneous polynomials of degree j and f_s is the first polynomial different from zero.

Let $\mathcal{U}_{s+\nu}$ be an ideal of the polynomials ring $\mathbf{C}[z_1, \dots, z_N]$ generated by $f_s, \dots, f_{s+\nu}$, $\nu = 0, 1, \dots$

Since $\mathcal{U}_s \subset \mathcal{U}_{s+1} \subset \dots$ then by the ascending chain condition ([5]) there exists an integer l such that

$$\mathcal{U}_s \subset \mathcal{U}_{s+1} \subset \dots \subset \mathcal{U}_{s+l-1} \subsetneq \mathcal{U}_{s+l} = \mathcal{U}_{s+l+1} = \dots$$

Let us take integers $s = s_0 < s_1 < \dots < s_k = s+l$ such that

$$\mathcal{U}_{s0} \subsetneq \mathcal{U}_{s_1} \subsetneq \dots \subsetneq \mathcal{U}_{s_k}$$

and

$$\mathcal{U}_{s_j} = \mathcal{U}_{s_j+\nu} \quad \text{for} \quad \nu = 0, 1, \dots, s_{j+1} - s_j - 1,\ j = 0, \dots, k-1 .$$

Homogeneous polynomials $F_j := f_{s_j}$, $j = 0, \dots, k$, form a homogeneous basis of an ideal $\mathcal{U}_f$, where $\mathcal{U}_f$ denotes the smalest ideal of $\mathbf{C}[z_1, \dots, z_N]$ containing homogeneous polynomials $f_s, f_{s+1}, \dots$ Hence there exist polynomials $g_0^{(j)}, \dots, g_k^{(j)}$ such that

$$f_j = F_0 g_0^{(j)} + \dots + F_k g_k^{(j)}, \quad j = s, s+1, \dots \tag{2}$$

Presenting $f \circ \chi_w$ in the form

$$f \circ \chi_w(\lambda) = \sum_{j=s}^{\infty} f_j(w) \lambda^j$$

and applying (2) we get

$$k(w) = \begin{cases} 0 & \text{for } w \in Z(F_0) \cap Z(F_1) \cap \dots \cap Z(F_k) \\ s & \text{for } w \notin Z(F_0) \\ s_j & \text{for } w \in \big(Z(F_0) \cap \dots \cap Z(F_{j-1})\big) - Z(F_j) \end{cases}$$

and hence

$$0 \leqslant k(w) \leqslant s_k = s+1 .$$

Example. Let $N = 3$ and

$$f(z_1, z_2, z_3) = z_1 + z_2^2 + z_3^3 + \sum_{j \geqslant 4} f_j(z_1, z_2, z_3),$$

where f_j are homogeneous polynomials of degree j such that the function f is entire. In this case

$$\mathcal{U}_1 \subsetneq \mathcal{U}_2 \subsetneq \mathcal{U}_3 = \mathcal{U}_4 = \dots$$

Indeed, $\mathscr{U}_3$ contains all the homogeneous polynomials of a degree greater than 3. Hence

$$k(w_1, w_2, w_3) = \begin{cases} 1 \text{ when } w_1 \neq 0 \\ 2 \text{ when } w_1 = 0, w_2 \neq 0 \\ 3 \text{ when } w_1 = w_2 = 0 \end{cases}$$

Remark. By analogy we may define the order of the zero of a germ of analytic functions. In this case an analogical theorem might be deduced from the proof of Theorem 1.

We call

$$\varkappa(z) = \sup\{k(z, w) : w \in \mathbf{C}^N - \{0\}\}$$

the upper order of the zero z of f. If $f(z) \neq 0$ then by definition $\varkappa(z) = 0$.

THEOREM 2. *The function $\varkappa$ is locally bounded in $\mathbf{C}^N$. Moreover, if for every $w \in \mathbf{C}^N - \{0\}$ $f \circ \varkappa_w \neq 0$ then inequality $\varkappa(\mathring{z}) \leqslant l$ implies $\varkappa(z) \leqslant l$ in a neighbourhood U of $\mathring{z}$.*

Proof. Consider an entire function

$$\Phi(z, w) := \sum_{j=0}^{\infty} \Big(\sum_{|\alpha|=j} \frac{D^\alpha f(z)}{\alpha!} w^\alpha \Big) = \sum_{j=0}^{\infty} \Phi_j(z, w)$$

of $2N$ complex variables (z, w), where

$$\alpha = (\alpha_1, \ldots, \alpha_N) \in N^N, |\alpha| = \alpha_1 + \ldots + \alpha_N, \alpha! = \alpha_1! \ldots \alpha_N!.$$

It is clear that

$$\Phi(\mathring{z}, z - \mathring{z}) = f(z) = \Phi(z, 0)$$

and

$$f(\mathring{z} + \lambda w) = \Phi(\mathring{z}, \lambda w) = \sum_{j=1}^{\infty} \Phi_j(\mathring{z}, w) \lambda^j.$$

Now fix $\mathring{z} \in \mathbf{C}^N$ and let w vary over all points of the sphere $S(r)$ of radius $r > 0$ about the point zero, and observe that $\varkappa(\mathring{z}) \leqslant l$ if and only if for every $w \in S(r)$

(3) $$\sum_{j \leqslant 1} |\Phi_j(\mathring{z}, w)| \neq 0 \quad \text{or} \quad 0 = \Phi_1(\mathring{z}, w) = \Phi_2(\mathring{z}, w) = \ldots$$

The local analysis of analytic sets (as described in Chapter II of [2]) gives the following information.

There are neighbourhoods U of $\mathring{z}$ in $\mathbf{C}^N$ and V of 0 in $\mathbf{C}^N$ and an integer k such that every Φ_j may be written in the form

(4) $$\Phi_j = \Phi_1 g_1^{(j)} + \ldots + \Phi_k g_k^{(j)}, \quad j = 1, 2, \ldots,$$

where $g_1^{(j)}, \ldots, g_k^{(j)}$ are holomorphic functions in $U \times V$. Therefore taking a small enough $r > 0$ and applying (3) we observe that $\varkappa(z) \leqslant k$ in U.

If $\varkappa(\mathring{z}) \leqslant k$ and for every $w \in \mathbf{C}^N - \{0\}$ $f \circ \chi_w \neq 0$ then

$$\inf\{\sum_{j \leqslant l} |\Phi_j(\mathring{z}, w)| : \|w\| = 1\} = \delta > 0.$$

By standard reasoning there exists a neighbourhood U of $\overset{\circ}{z}$ such that

$$\sum_{j\leq l} |\Phi_j(z, w)| \neq 0 \text{ in } U\times S(1),$$

thereby proving our theorem.

3. Remarks on Rudin's Theorem B. Keeping the notations of the proof of Theorem 1 we shall prove the following

THEOREM 3. *Let f be an entire function of form* (1). *If $\mathscr{U}_f$ denotes the ideal of $\mathbf{C}[z_1, \ldots, z_N]$ generated by polynomials $f_s, f_{s+1}, \ldots$, then $V = V(\mathscr{U}_f)$ is the maximal algebraic cone contained in $Z(f)$ and polynomials $F_0, \ldots, F_k$ have the following properties*:

1° *Polynomials $F_0, \ldots, F_k$ form a homogeneous basis of the ideal $\mathscr{U}_f$.*

2° *There exist entire functions $g_0, \ldots, g_k$ such that*

$$f = F_0 g_0 + \ldots + F_k g_k \text{ and } g_j(0) = 1, \quad j = 0, \ldots, k. \tag{5}$$

3° *There are polynomials $p_0^{(\nu)}, \ldots, p_k^{(\nu)}$ in $\mathbf{C}^N$ such that $p_l^{(\nu)}(0) = 1$, $l = 0, \ldots, k$, $\nu = 1, 2, \ldots$ and the sequence*

$$P_\nu = F_0 p_0^{(\nu)} + \ldots + F_k p_k^{(\nu)}, \ \nu = 1, 2, \ldots$$

converges to f uniformly on compact subsets in $\mathbf{C}^N$.

Proof. Property 1° is a trivial consequence of the proof of Theorem 1. Therefore, consider the property 2°. Because of (2), f lies in the closure of the ideal $I(F_0, \ldots, F_k)$ of $H(\mathbf{C}^N)$ generated by $F_0, \ldots, F_k$. Applying Rudin's Theorem A we get $f \in I(F_0, \ldots, F_k)$. Therefore, there are entire functions $g_0, \ldots, g_k$ such that

$$f = F_0 g_0 + \ldots + F_k g_k.$$

In order to prove 2° let us develop the functions g_j into the series

$$g_j = \sum_{l=0}^{\infty} g_{jl}, \quad j = 0, \ldots, k, \tag{6}$$

where g_{jl} are homogeneous polynomials in $\mathbf{C}^N$ of degree l. Now, from (5) we conclude that

$$F_j(1-g_{j0}) = F_0 g_{0(s_j-s_0)} + F_1 g_{1(s_j-s_1)} + \ldots + F_{j-1} g_{(j-1)(s_j-s_{j-1})}, \quad j = 0, \ldots, k. \tag{7}$$

Suppose that $g_{j0} \neq 1$. Then $F_j \in \mathscr{U}_{s_j-1}$ and this contradicts the definition of F_j. The property 3° is an immediate consequence of 2°.

In (5) the functions $g_0, \ldots, g_k$ are not univocally determined. Let for example $f(z_1, z_2, z_3) = z_1 z_2 + z_2^2 z_3$. In this case $F_0(z) = z_1 z_2$, $F_1(z) = z_2^2 z_3$. Because

$$f(z) = F_0(z) + F_1(z) \text{ and } f(z) = F_0(z)(1+z_2 z_3) + F_1(z)(1-z_1)$$

we obtain two different decomposition of f.

Let $\mathscr{U}$ be a homogeneous ideal of $\mathbf{C}[z_1, \ldots, z_N]$. Then $V = V(\mathscr{U})$ is the algebraic cone in $\mathbf{C}^N$. Using the fact that the ring $C[z_1, \ldots, z_N]$ is Noetherian we can find universal generators $\tilde{F}_0, \ldots, \tilde{F}_k$ of the ideal $\mathscr{U}(V)$. Let the function $f = f_s + f_{s+1} + \ldots$ be entire in $\mathbf{C}^N$ and let $V \subset Z(f)$. Ideal $I(V)$ (of $H(\mathbf{C}^N)$) is the homogeneous ideal. Then all homogeneous polynomials $f_s, f_{s+1}, \ldots$ belong to $I(V)$. Therefore

$$S_\nu = f_s + f_{s+1} + \ldots + f_{s+\nu} \in I(V), \quad \nu = 0, 1, \ldots$$

Hence

$$S_\nu \in \mathcal{U}(\tilde{F}_0, \ldots, \tilde{F}_k) \subset I(\tilde{F}_0, \ldots, \tilde{F}_k) .$$

Now, applying Rudin's Theorem A we obtain $f \in I(F_0, \ldots, F_k)$. Consequently

$$f = \tilde{F}_0 \tilde{g}_0 + \ldots + \tilde{F}_k \tilde{g}_k ,$$

where $\tilde{g}_0, \ldots, \tilde{g}_k$ are entire functions which on the contrary to Theorem 3 may vanish at 0.

The non-homogeneous case is a direct consequence of the homogeneous case and of the following.

THEOREM 4 ([4], proof of Theorem 4.1). *Suppose V is an algebraic variety in $\mathbf{C}^N$, f is an entire function in $\mathbf{C}^N$, and $V \subset Z(f)$. Then there exists an entire function F of $N+1$ complex variables $(z_1, \ldots, z_N, \zeta) = (z, \zeta)$ such that*

1° $F(z, 1) = f(z)$, $z \in \mathbf{C}^N$,

2° $\{(\zeta z, \zeta) \in \mathbf{C}^{N+1} \colon z \in V, \zeta \in \mathbf{C}\} \subset Z(F)$.

The proof of Theorem 4 is based on the fact that the sheaf of germs of holomorphic functions in $\mathbf{C}^{N+1}$ which vanish on the analytic variety in $\mathbf{C}^{N+1}$ is coherent ([1], [2]) and on Cartan's Theorem B ([2], [3]).

References

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