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On maximal subgroups in semigroups of partial transformations

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1. The general construction of all maximal subgroups for an arbitrary given semigroup is known (see [1], pp. 41–43). In the present note we apply this construction to the semigroup of all partial transformations in a given set X. In this paper for any function f the symbols D_f and $\mathcal{A}_f$ denote its domain and its set of values respectively. For the arbitrarily given functions f, g we use their "relational" superposition fg in the following sense

$$(1)\qquad \begin{cases} D_{fg} = g^{-1}(\mathcal{A}_g \cap D_f), \\ \bigwedge\limits_{x \in D_{fg}} (fg)(x) = f(g(x)). \end{cases}$$

2. Let X be an arbitrary nonempty set. The set S of all partial transformations in X (including the empty transformation $\emptyset$) with the above-mentioned relational superposition as binary operation is a semigroup, which is called the semigroup of partial transformations in X.

One can easily verify that $\varphi \in S$ is an idempotent in this semigroup if and only if

$$(2)\qquad \mathcal{A}_\varphi \subset D_\varphi \wedge \mathcal{A}_\varphi = \{x \in D_\varphi \colon \varphi(x) = x\}.$$

The transformation $\emptyset$ is evidently an idempotent in S.

3. The unit of an arbitrary subgroup of S is an idempotent of S. Our problem is to determine the maximal subgroups of S for all idempotents of S as their units. It is easily seen that $\emptyset$ is the unit in the maximal subgroup $\{\emptyset\}$ of S. For the sequel we assume that φ is a nonempty idempotent of S.

3.1. First we determine the greatest subsemigroup S^* of S for which φ is a right unit.

$$S^* = \{f \in S \colon f\varphi = f =^{(1)} \{g\varphi \colon g \in S\}.$$

We have the following

LEMMA 1. *For every $f \in S$, f is an element of S^* if and only if*

$$(3)\qquad D_f \subset D_\varphi \wedge \bigwedge_{x_1, x_2 \in D_f} \varphi(x_1) = \varphi(x_2) \Rightarrow f(x_1) = f(x_2) \wedge$$
$$\wedge \bigwedge_{u \in X} \varphi^{-1}(\{u\}) \cap D_f \neq \emptyset \Rightarrow \varphi^{-1}(\{u\}) \subset D_f.$$

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(1) This equality and the subsemigroup character of S^* are well known for each semigroup S and its idempotent φ (see [1], p. 42).

Indeed if $f = g\varphi$, where $g \in S$, then $D_f \subset D_\varphi$ and for $x_1, x_2 \in D_f$ from the equality $\varphi(x_1) = \varphi(x_2)$ it follows that $f(x_1) = f(x_2)$. Furthermore if $x_0, u \in X$ and

$$x_0 \in \varphi^{-1}(\{u\}) \cap D_f$$

then $x_0 \in D_f = D_{g\varphi}$. Using (1) we obtain $\varphi(x_0) = u \in D_g$ and $g(u) = f(x_0)$. For the above given u and an arbitrary $x \in \varphi^{-1}(\{u\})$ there is $\varphi(x) = u$ and by $u \in D_g$ the relation $x \in D_{g\varphi} = D_f$ holds. Thus if $\varphi^{-1}(\{u\}) \cap D_f \neq \emptyset$ then $\varphi^{-1}(\{u\}) \subset D_f$.

Conversely let $f \in S$ fulfil (3). We introduce the transformation g as follows:

$$D_g := \mathcal{A}_\varphi$$

and for $u \in \mathcal{A}_\varphi$ when $\varphi^{-1}(\{u\}) \cap D_f = \emptyset$ then $g(u)$ is an arbitrarily chosen element of X; when $x \in \varphi^{-1}(\{u\}) \cap D_f$ then $g(u) = f(x)$.

Using (3) we find immediately that g is a well defined element of S and $f = g\varphi$.

Next we find the greatest subsemigroup $*S$ with a left unit φ in S. Obviously there is

$$*S = \{f \in S\colon \varphi f = f\} = \{\varphi g\colon g \in S\}.$$

It is easily seen from (1) that we have

LEMMA 2. *For every $f \in S$, f belongs to $*S$ if and only if*

$$\mathcal{A}_f \subset \mathcal{A}_\varphi. \tag{4}$$

The intersection $S_\varphi := *S \cap S*$ is the greatest subsemigroup of S with the two-sided unit φ.

3.2. The maximal subgroup with the unit φ of the semigroup S is the subset G_φ of S such that

$$f \in G_\varphi \Leftrightarrow f \in S_\varphi \wedge \bigvee_{g \in S_\varphi} fg = gf = \varphi. \tag{5}$$

To determine all the elements of G_φ first observe that if $f, g \in S_\varphi$ and $gf = \varphi$ then $D_\varphi \subset D_f$ and consequently $D_f = D_\varphi$. Analogously the equality $fg = \varphi$ for $f, g \in S_\varphi$ implies that $\mathcal{A}_\varphi \subset \mathcal{A}_f$ and further $\mathcal{A}_\varphi = \mathcal{A}_f$.

Thus we have the following

LEMMA 3. *If $f \in G_\varphi$ then*

(6) *f is a mapping of D_φ into itself and $\mathcal{A}_f = \mathcal{A}_\varphi$.*

Moreover if $f, g \in S_\varphi$ and $gf = \varphi$ then $\varphi^{-1}\varphi = f^{-1}g^{-1}gf$ (here we have the superpositions of binary relations in X). From $i_{D_\varphi} \subset g^{-1}g$ it follows that $f^{-1}f = f^{-1}i_{D_\varphi}f \subset f^{-1}g^{-1}gf = \varphi^{-1}\varphi$. Using (3) from this inclusion we obtain

LEMMA 4. *If $f \in G_\varphi$ then*

$$f^{-1}f = \varphi^{-1}\varphi. \tag{7}$$

Finally from (2) and (7) for every $f \in G_\varphi$ and $y_1, y_2 \in \mathcal{A}_\varphi = \mathcal{A}_f$ there is

$$f(y_1) = f(y_2) \Rightarrow \varphi(y_1) = \varphi(y_2) \Rightarrow y_1 = y_2.$$

Furthermore if $f \in G_\varphi$ and $y \in \mathcal{A}_\varphi$ then $y = \varphi(y) = f(g(y))$ where g is the inverse element of f in G_φ. Thus we have

LEMMA 5. *If* $f \in G_\varphi$ *then*

(8) $f \mid \mathfrak{A}_\varphi$ *is a permutation of* $\mathfrak{A}_\varphi$.

Consider now the set

$$H_\varphi := \{ f \in S\colon\ D_f = D_\varphi \wedge \mathfrak{A}_f = \mathfrak{A}_\varphi \wedge \varphi^{-1}\varphi = f^{-1}f \wedge f \mid \mathfrak{A}_\varphi \text{ is a permutation of } \mathfrak{A}_\varphi \}.$$

This is Green's $\mathscr{H}$-class containing the idempotent φ in the semigroup formed of all mappings of D_φ into itself (with usual superposition) and thus it is a subgroup of this semigroup (see [1], pages 79, 81, 91).

This completes the proof of the following.

THEOREM. *Let* φ *be a nonempty idempotent of the semigroup* S *of all partial transformations in* X. *The maximal subgroup* G_φ *of* S *with the unit* φ *is defined by the following formula*:

$$G_\varphi := \{ f \in S\colon\ D_f = D_\varphi \wedge \mathfrak{A}_f = \mathfrak{A}_\varphi \wedge f^{-1}f = \varphi^{-1}\varphi \wedge f \mid \mathfrak{A}_\varphi \text{ is a permutation of } \mathfrak{A}_\varphi \}.$$

4. COROLLARY. *Let* f *be a function*, $f\colon A \to B$, $A \neq \emptyset$. *There exists a group* F *of functions with the superposition understood as in* (1), *containing* f *if and only if*

$$\mathfrak{A}_f \subset A \wedge f \mid \mathfrak{A}_f \text{ is a permutation of } \mathfrak{A}_f. \tag{9}$$

Indeed if such a group F exists let X be the union of domains and sets of values of all functions belonging to F. Then F is a subgroup of the semigroup of all partial transformations in X. If φ is the unit in F then $F \subset G_\varphi$, which proves (9).

Conversely if for f (9) holds true then we form the semigroup of all partial transformations in A (which contains f) and we define its idempotent φ putting for

$$u \in \mathfrak{A}_f, \quad \varphi(f^{-1}(\{u\})) := \{u\}.$$

It is evident that f and φ satisfy the conditions (6), (7), (8) and $f \in G_\varphi$.

Remark: The result of this corollary was first proved by Z. Moszner [2] by using the theory of the translation equation.

References

[1] А. Клиффорд, Г. Престон *Алгебраическая теория полугрупп*, т.I, Москва, 1972,

[2] Z. Moszner, *Sur les groupes de fonctions* (in printing), Ann. Polon. Math. t. XXXVII.

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