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Solution of the translation equation on some structures

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1. In this paper we're giving general solution of the translation equation on three types of semigroups (semigroups with zero-multiplication, semigroups of left units, semigroups of right units) and on arbitrary structures with adjoint units. Let $(S, \cdot)$ be a structure (where S is a nonempty set and "$\cdot$" is a binary operation in S defined for every $(x, y) \in S \times S$) and M an arbitrary nonempty set.

The functional equation

$$\text{(T)} \qquad F(F(x, a), b) = F(x, a \cdot b)$$

with unknown function $F\colon M \times S \to M$ is called *the translation equation on* $(S, \cdot)$ (*with the fibre* M).

An arbitrary given function $F\colon M \times S \to M$ fulfilling the condition

$$\bigwedge_{x \in M} \bigwedge_{a,b \in S} F(F(x, a), b) = F(x, a \cdot b)$$

is called *a solution of the equation* (T) *on* $(S, \cdot)$. The set of all such functions is called *the general solution of* (T) *on* $(S, \cdot)$.

Every family $(M_i)_{i \in I}$ of nonempty and disjoint sets such that $\bigcup_{i \in I} M_i = M$ is called *a partition of* M. We suppose in the sequel that if $(M_i)_{i \in I}$ is a partition of M then

$$M_i \cap M_j = \emptyset \quad \text{for } i \neq j,\ i, j \in I.$$

2. Let $(S, \cdot)$ be a semigroup such that $\operatorname{card} S \cdot S = 1$. This unique element of $S \cdot S$ we denote by 0. Such a semigroup is called *a semigroup with zero-multiplication.*

We have the following

THEOREM 1. *A mapping $F\colon M \times S \to M$ is a solution of* (T) *on* $(S, \cdot)$ *if and only if it is constructed in the following way*

1° *We take an arbitrary partition $(M_i)_{i \in I}$ of M.*

2° *We choose, for $i \in I$, an arbitrary subset P_i of M_i and fixed element $x_i \in P_i$.*

3° *For every $i \in I$ we introduce a mapping f_i of the product set $M_i \times S$ onto P_i such that*

$$f_i(x, a) = x_i \quad \textit{for } x \in P_i \quad \textit{or } a = 0.$$

4° *We put*

$$F(x, a) := f_i(x, a) \quad \textit{for } x \in M_i .$$

Proof. Suppose that F is a solution of the translation equation and consider the function f: $M \to M$ such that

$$f(x) := F(x, 0) \quad \text{for } x \in M .$$

For each $i \in f(M)$ we form $M_i := f^{-1}(\{i\})$. It is clear that the family $(M_i)_{i \in f(M)}$ is a partition of M.

Next, from

$$f(F(x, a)) = F(F(x, a), 0) = F(x, 0) = f(x) = i$$

for all $x \in M_i$ and $a \in S$, we have $F(M_i \times S) \subset M_i$ and $i \in F(M_i \times S)$.

Moreover we obtain

$$F(F(x, a), b) = F(x, a \cdot b) = F(x, 0) = f(x) = i .$$

Thus we have shown that $f_i := F|_{M_i \times S}$ satisfies the condition stated in the theorem with $P_i := F(M_i \times S)$ and $x_i := i$. We have also $F = \bigcup_{i \in f(M)} f_i$, whence the proof of necessity is complete.

Conversely, suppose that a mapping F is constructed by 1°, 2°, 3° and 4°. It is evident that 3° implies

$$F(F(x, a), b) = x_i = F(x, 0) \quad \text{for } x \in M_i,\ a, b \in S .$$

Therefore the equality

$$F(F(x, a), b) = F(x, a \cdot b)$$

holds for all $x \in M$ and $a, b \in S$.

3. Let $(S, \cdot)$ be a semigroup such that $\bigwedge_{a,b \in S} a \cdot b = b$. It is called *a semigroup of left units.*

We have the following

THEOREM 2. *A mapping F: $M \times S \to M$ is a solution of* (T) *on $(S, \cdot)$ if and only if it is constructed as follows*

5° *We take a partition $(M_i)_{i \in I}$ of M.*

6° *We denote by $\mathscr{F}$ the set of all functions f: $M \to M$ such that*

$$\bigwedge_{i \in I} \left(f(M_i) \subset M_i \wedge \operatorname{card} f(M_i) = 1 \right) .$$

7° *We take an arbitrary function φ: $S \to \mathscr{F}$.*

8° *We define $F(x, a) := (\varphi(a))(x)$ for $(x, a) \in M \times S$.*

Proof. Let F: $M \times S \to M$ be a solution of (T) on $(S, \cdot)$. This means that

9° $\bigwedge_{x \in M} \bigwedge_{a,b \in S} F(F(x, a), b) = F(x, b) .$

We define f_a: $M \ni x \mapsto F(x, a)$ for $a \in S$ and

$$\varphi: S \ni a \mapsto f_a \in M^M.$$

For every $a, b \in S$ from 9° we obtain the equalities

$$f_a f_a = f_a, \quad f_a f_b = f_a, \quad f_b f_a = f_b.$$

Therefore by evident inclusions $i_M \subset f_a^{-1} f_a$, $i_M \subset f_b^{-1} f_b$ we have

$$f_a^{-1} f_a = f_a^{-1} i_M f_a \subset f_a^{-1} f_b^{-1} f_b f_a = f_b^{-1} f_b,$$
$$f_b^{-1} f_b \subset f_b^{-1} f_a^{-1} f_a f_b = f_a^{-1} f_a,$$

whence $f_a^{-1} f_a = f_b^{-1} f_b$. This means that $M/f_a^{-1} f_a = M/f_b^{-1} f_b$ for every $a, b \in S$. Let $(M_i)_{i \in I}$ denote the common partition $M/f_a^{-1} f_a$ of M. For $i \in I$ and $a \in S$ from the definition of M_i there is card $f_a(M_i) = 1$ and $f_a(M_i) \subset M_i$,because f_a is an idempotent of the semigroup M^M. Therefore if $\mathscr{F}$ has the same sense as in 6°, then there is evidently

$$\varphi: S \dot{\rightarrow} \mathscr{F}.$$

Furthermore for $(x, a) \in M \times S$ we have $F(x, a) = (\varphi(a))(x)$, which completes the proof of necessity.

Conversely if F is constructed as in 5°, 6°, 7°, 8° then for $x \in M$, $a, b \in S$ there is $x \in M_i$, where $i \in I$ is uniquely chosen, and $(\varphi(a))(x) \in M_i$. Moreover $(\varphi(b))((\varphi(a))(x)) = (\varphi(b))(x)$, because $\operatorname{card}(\varphi(b))(M_i) = 1$.

$$F(F(x, a), b) = (\varphi(b))((\varphi(a))(x)) = (\varphi(b))(x) = F(x, b) = F(x, a \cdot b)$$

and our theorem has been proved.

4. Let $(S, \cdot)$ be a semigroup such that $\bigwedge_{a,b \in S} a \cdot b = a$. It is called *a semigroup of right units.*

The general solution of (T) on $(S, \cdot)$ we obtain using

THEOREM 3. *A mapping F: $M \times S \to M$ is a solution of* (T) *on* $(S, \cdot)$ *if and only if it is constructed as follows*

10° *We take an arbitrary subset V of M, $V \neq \emptyset$.*

11° *We denote by $\mathscr{F}$ the set of all functions f: $M \to M$ such that $f(M) = V$ and $f|_V = i_V$.*

12° *We take a function φ: $S \to \mathscr{F}$.*

13° *We define $F(x, a) = (\varphi(a))(x)$ for $(x, a) \in M \times S$.*

Proof. Suppose that F: $M \times S \to M$ is a solution of (T) on $(S, \cdot)$ which means that

14° $\bigwedge_{x \in M} \bigwedge_{a,b \in S} F(F(x, a), b) = F(x, a)$.

Let us define $V := F(M \times S)$ and

$$\varphi: S \ni a \mapsto f_a = F(\cdot, a) \in M^M.$$

Evidently $F(x, a) = (\varphi(a))(x)$, for $(x, a) \in M \times S$. We have also $F(x, s) = F(F(x, s), a)$ for $(x, s) \in M \times S$. Hence $f_a(M) = V$, for every $a \in S$.

Furthermore from 14° we conclude that $f_a f_a = f_a$ and thus

$$f_a|_V = i_V \quad \text{for } a \in S .$$

Conversely let F be constructed as in 10°, 11°, 12°, 13°. If $x \in M, a, b \in S$ then $F(F(x,a),b) = (\varphi(b))((\varphi(a))(x))$. But $(\varphi(a))(x) \in V$ and $\varphi(b)|_V = i_V$, thus

$$F(F(x,a),b) = (\varphi(a))(x) = F(x,a) = F(x, a\cdot b)$$

and the proof of our theorem is complete.

Remark 1. Obviously theorems 1, 2, 3 give a method which allows the construction of all subsemigroups of M^M, which are semigroups with zero-multiplication, semigroups of left units and semigroups of right units respectively.

5. Let $(S,\cdot)$ be an arbitrary structure. We denote by $(S^1,\cdot)$ the structure obtained from $(S,\cdot)$ by adjoining a unit $1 \notin S$.

Remark 2. A function $F\colon M\times\{1\}\to M$ satisfies the translation equation on $(\{1\},\cdot)$ if and only if $F(\cdot\,,1)$ is an idempotent of the semigroup M^M.

THEOREM 4. *Let $F_1\colon M\times S\to M$ and $F_2\colon M\times\{1\}\to M$ satisfy the translation equation on $(S,\cdot)$ and on $(\{1\},\cdot)$ respectively. A mapping $F\colon M\times S^1\to M$ defined by $F := F_1\cup F_2$ is a solution of the translation equation on $(S^1,\cdot)$ if and only if*

$$(*) \qquad F_1(M\times S)\subset F_2(M\times\{1\}) \quad \textit{and} \quad \bigwedge_{x\in M}\bigwedge_{a\in S} F_1(x,a) = F_1(F_2(x,1),a) .$$

Proof. Suppose that $F := F_1 \cup F_2$ satisfies the equation of translation. Then for all $x\in M$, $a\in S$ we have

$$F_1(x,a) = F(x,a) = F(F(x,a),1) = F_2(F(x,a),1)$$

and

$$F_1(F_2(x,1),a) = F(F(x,1),a) = F(x,a) = F_1(x,a) .$$

Thus $(*)$ holds.

Conversely, if F_1, F_2 fulfil the supposition of the theorem and the condition $(*)$ holds, then we obtain for $x\in M$ and $a\in S$

$$F(F(x,a),1) = F_2(F_1(x,a),1) = F_1(x,a) = F(x,a) = F(x, a\cdot 1)$$

because $F_2(\cdot\,,1)$ is an idempotent of M^M and

$$F_2(\cdot\,,1)|_{F_2(M\times\{1\})} = i_{F_2(M\times\{1\})} ,$$

Moreover

$$F(F(x,1),a) = F_1(F_2(x,1),a) = F_1(x,a) = F(x,a) = F(x, a\cdot 1)$$

and thus F is a solution of (T) on $(S^1,\cdot)$.

COROLLARY 1. *Every function f which is an idempotent of M^M and satisfies $(*)$, where $F_1\colon M\times S\to M$ and for $x\in M$ $F_2(x,1) = f(x)$, is constructed in the following way*

15° *We consider the family* $(M_i)_{i\in I}$ *of all equivalence classes of the following relation in* M

$$\{(x, y) \in M\times M\colon \bigwedge_{a\in S} F_1(x, a) = F_1(y, a)\}\,.$$

16° *We take arbitrary idempotents* f_i *of* $M_i^{M_i}$ *(for* $i\in I$*) such that*

$$F_1(M\times S)\cap M_i \subset f_i(M_i)\,.$$

17° *We define* $f := \bigcup_{i\in I} f_i$.

Indeed, if a mapping f is formed according to 15°, 16° and 17°, then $\bigcup_{i\in I} f_i(M_i) = f(M)$ and F_1, F_2 have the property $(*)$. Conversely, if an idempotent f satisfies $(*)$ we put for $i\in I$

$$f_i := f|_{M_i}\,.$$

This implies

COROLLARY 2. *Let* $F_1\colon M\times S\to M$ *satisfy the translation equation on* $(S,\cdot)$. *A mapping* $F\colon M\times S^1\to M$ *is a solution of this equation on* $(S^1,\cdot)$ *which is an extension of* F_1 *if and only if a mapping* $f\colon M\ni x\mapsto F(x, 1)\in M$ *is built as in* 15°, 16° *and* 17°.

This extension of the solution F_1 *is unique if and only if* $\bigwedge_{i\in I}$ card $M_i>1\Rightarrow M_i\subset \mathrm{F}_1(M\times S)$ *and then* $f = i_M$.

Consequently, every solution of (T) *on* $(S,\cdot)$ *can be extended, but this extension, in general, is not unique.*

On the other hand we have

THEOREM 5. *Let* $f\colon M\to M$ *be such that* $f(f(x)) = f(x)$ *for all* $x\in M$ *and let* $F_2(x, 1) := f(x)$. *A mapping* $F\colon M\times S^1\to M$ *is a solution of the translation equation on* $(S^1,\cdot)$ *which is an extension of* F_2 *if and only if there exists the function* $\tilde F\colon f(M)\times S\to f(M)$ *fulfilling the equation of translation on* $(S,\cdot)$ *(with the fibre* $f(M)$*) and such that*

$$F(x, a) = \tilde F(f(x), a) \quad \text{for } x\in M \text{ and } a\in S\,.$$

Proof. Suppose that a mapping F fulfils the condition stated in the theorem. Since $\tilde F(f(x), a)\in f(M)$ for an arbitrary $x\in M$, $a\in S$ and f is an idempotent of M^M, we obtain

$$\begin{aligned} F(F(x, a), b) = F\big(\tilde F(f(x), a), b\big) &= \tilde F\Big(f\big(\tilde F(f(x), a)\big), b\Big) \\ &= \tilde F\big(\tilde F(f(x), a), b\big) = \tilde F(f(x), a\cdot b) = F(x, a\cdot b)\,. \end{aligned}$$

We have also

$$\begin{aligned} F\big(F(x, 1), a\big) &= F\big(F_2(x, 1), a\big) = F\big(f(x), a\big) = \tilde F\big(f(f(x)), a\big) = \tilde F(f(x), a) \\ &= F(x, a) = \tilde F(f(x), a) = f\big(\tilde F(f(x)), a\big) = F_2\big(\tilde F(f(x), a), 1\big) = F\big(F(x, a), 1\big)\,. \end{aligned}$$

Thus F is a solution of (T) on $(S^1,\cdot)$.

Now suppose that $F\colon M\times S^1\to M$ satisfies the translation equation on $(S^1,\cdot)$ where $f(x) = F(x, 1)$ for $x\in M$ and let $\tilde F := F|_{f(M)\times S}$. Then $F|_{M\times S}$ and $\hat F$ satisfy the translation equation on $(S, \cdot)$ (but $\hat{F}$ with the fibre $f(M)$). For the mapping $F|_{M\times S}$ we have from theorem 4 that

$$F(f(x), a) = F(x, a) \quad \text{for } x \in M,\ a \in S.$$

This yields

$$\tilde{F}(f(x), a) = F(f(x), a) = F(x, a) \quad \text{for } x \in M,\ a \in S.$$

Thus our theorem is proved.

COROLLARY 3. *Every idempotent $f\colon M \to M$ can be extended on the set $M \times S^1$ to a solution of* (T) on $(S^1, \cdot)$, *but in general not uniquely.*

This actual extension is unique if and only if $\operatorname{card} f(M) = 1$.

In particular, $f = i_M$ can be stuck together with every solution F_1 of the equation of translation on $(S, \cdot)$ giving the function F which satisfies the equation of translation on $(S^1, \cdot)$ and f is the unique idempotent of M^M, satisfying this condition.

However, a solution $F\colon M \times S \to M$ of (T) *can be stuck together with every idempotent $f \in M^M$ if and only if* $\operatorname{card} M = 1$.

A solution of the equation of translation on a structure $(S, \cdot)$ with adjoint zero has been given by Z. Moszner and J. Tabor in [1].

References

[1] Z. Moszner et J. Tabor, *L'équation de translation sur une structure avec zéro*, Ann. Pol. Math. XXXI 1976, 255–264.

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