ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DXXX PRACE MATEMATYCZNE, ZESZYT 21 1979

On zero points of the linear combinations of the eigenfunctions in a singular Sturm–Liouville problem

by S. Postawa

In 1962 the paper [1] appeared, in which the author considered the problem of the oscillation of linear combinations of eigenfunctions the regular Sturm–Liouville problem, i.e. that of the bounded interval $[a, b]$. The present note is an attempt to transfer the results of the paper [1] to just one particular case of the Sturm–Liouville problem, i.e. the case of the interval $[a, +\infty)$.

§ 1. Let us consider the problem of the eigenvalues and eigenfunctions for the equation:

(1) $$L[u]+\lambda\varrho(x)u = 0,$$

where

(2) $$L[u] = [p(x)u'(x)]'-q(x)u(x)$$

with the boundary condition,

(3) $$\alpha_1 u(a)-\alpha_2 u'(a) = 0.$$

Let us replace the right boundary condition by the following:

(4) $$u\in\mathscr{L}_2([a, +\infty)) \quad\text{and}\quad u'\in\mathscr{L}_2([a, +\infty)).$$

We assume that α_1, α_2 are real non-negative numbers fulfilling the condition: $\alpha_1^2+\alpha_2^2>0$. Suppose that the functions $\varrho(x)>0$ and $p(x)>0$ in $[a, +\infty)$ and $p\in\mathscr{L}_2([a, +\infty))$ and $p\in C^1((a, +\infty))$, $\varrho\in\mathscr{L}_2([a, +\infty))$ and $\varrho\in C((a, +\infty))$, $q(x)\geq 0$, q is continuous in the interval $[a, +\infty)$ and $\lim_{x\to\infty} q(x) = +\infty$.

Moreover, suppose that the following holds:

Assumption Z. For the problem (1), (3), (4), there exists a sequence of eigenvalues

(5) $$0\leq\lambda_1<\lambda_2<\lambda_3\ldots, \lim_{n\to\infty}\lambda_n = +\infty$$

and a sequence of eigenfunctions of classes $\mathscr{L}_2$ and C^2 in $[a, +\infty)$

(6) $$u_1(x), u_2(x), u_3(x), \ldots$$

such that $u_n\in\mathscr{L}_2([a, +\infty))\cap C^2((a, +\infty))$ for every $n = 1, 2, 3, \ldots$ (see [2], Chapter V).

Assuming that $\lim_{x\to\infty} q(x) = +\infty$, the n-th eigenfunction $u_n(x)$ has exactly $n-1$ zero points in the interval $(a, +\infty)$, which are not zero points of its derivative (see [2], p. 113).

In the present paper we shall give the proof of the following:

THEOREM 1. *Every linear combination of eigenfunctions* (6) *of the problem* (1), (3), (4), *of the form*

(7) $$f(x) = c_m u_m(x) + \dots + c_n u_n(x), \qquad n \geq m \geq 1,$$

$c_m, \dots, c_n$ *real constants*, $c_m^2 + \dots + c_n^2 > 0$, *has in the interval* $(a, +\infty)$ *at least* $m-1$, *and at most* $n-1$ *zero points.*

§ **2.** To prove this theorem we shall prove the following lemmas:

LEMMA 1. *If*:

1° $f(x)$ *and* $g(x) = p(x)f'(x)$ (*where* p — *continuous and positive in* $[a, +\infty)$) *are of class* $\mathcal{L}_2$ *in* $[a, +\infty)$,

2° $f(x)$ *fulfils the boundary condition* (3),

3° $f(x) \neq 0$ *in* $(a, +\infty)$,

then there exists a point $\eta \in (a, +\infty)$ *such that* $h(\eta) f(\eta) < 0$, *where*

(8) $$h(x) = L[f(x)].$$

Proof. Since $f \in \mathcal{L}_2([a, +\infty))$, hence $f(x) \to 0$ for $x \to \infty$. Assuming that $f(x) > 0$, then $f(x)$ in the point a fulfils one of the following conditions: 1° $f(a) = 0$; 2° $f(a) > 0$ and $f'(a) > 0$; 3° $f(a) > 0$ and $f'(a) = 0$. It is easy to see that in every one of the cases 1°, 2°, 3° there exists a point $a \leq x_0 < +\infty$ such that $f'(x_0) = 0$ and $f(x_0) > 0$. By $f(x) \to 0$ for $x \to +\infty$ and $f(x_0) > 0$, there exists a point $x_0 < x_1 < \infty$, such that $f'(x_1) < 0$, or else $g(x_0) = p(x_0) f'(x_0) = 0$, $g(x_1) = p(x_1) f'(x_1) < 0$. Hence

$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} < 0.$$

But from the mean value theorem

$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} = g'(\eta),$$

hence $g'(\eta) < 0$ and $f(\eta) > 0$, because $x_0 < \eta < x_1$. We have then $h(\eta) = g'(\eta) - q(\eta) f(\eta) < 0$. The inequality $h(\eta) > 0$, when $f(x) < 0$ in (a, ∞), can be obtained arguing as above, in application to the function $-f(x)$, which gives $-h(\eta) < 0$, and this proves Lemma 1.

LEMMA 2. *If the functions* $f(x)$ *and* $g(x) = p(x) f'(x)$ (*where* p *is continuous and positive in* $[a, \infty)$) *are of the class* $\mathcal{L}_2$ *in* $[a, \infty)$, *and if for certain points* η_1, y, η_2 *of the interval* $(a, +\infty)$ *the relations* $\eta_1 < y < \eta_2$, $h(\eta_1) f(\eta_1) < 0$, $h(\eta_2) f(\eta_2) < 0$, $f(y) = 0$ *hold, then there exists at least one point* $\zeta \in (\eta_1, \eta_2)$ *such that* $h(\zeta) = 0$, *where* $h(x)$ *is defined by formula* (8).

The proof of Lemma 2 is to be found in [1].

LEMMA 3. *If:*

1° *$f(x)$ and $g(x) = p(x)f'(x)$ (p — continuous and positive in $[a, +\infty)$) are of the class $\mathscr{L}_2$ in $[a, +\infty)$;*

2° *f satisfies the boundary condition* (3);

3° *f has p zero points in $(a, +\infty)$, $1 \leqslant p \leqslant \infty$, then the function $h(x)$ defined by the formula* (8) *has at least p zero points in this interval.*

Proof. Let us assume for the present that $1 \leqslant p < \infty$, and $x_1, \ldots, x_p$ are the zero points of function $f(x)$ in $(a, +\infty)$. For intervals (x_i, x_{i+1}) $i = 0, 1, \ldots, p-1$, where $x_0 = a$ we are using Lemma 1 from [1], and for the interval $(x_p, +\infty)$ we are using Lemma 1 which is given in this paper. With these lemmas, we know that in every interval (x_i, x_{i+1}) $i = 0, 1, \ldots, p$, $x_0 = a$ and $(x_p, +\infty)$ exists a point η_i such that $h(\eta_i)f(\eta_i) < 0$ $(i = 0, 1, \ldots, p-1)$ and by Lemma 2 in every interval (η_i, η_{i+1}) $(i = 0, 1, \ldots, p-1)$ there exists at least one point ζ_i such that $h(\zeta_i) = 0$ $(i = 0, 1, \ldots, p-1)$ and this proves Lemma 3. If $p = \infty$, then the proof is to be found in [1].

LEMMA 4. *If:*

1° *the functions of the sequence $\{f_n(x)\}$ $(n = 1, 2, \ldots)$ are of the class C^1 in the interval $(a, +\infty)$, and are square integrable in the interval $[a, +\infty)$,*

2° *for each $b \in (a, +\infty)$ $f_n(x)$ tends uniformly to $f(x)$ and $f_n'(x)$ tends uniformly to $f'(x)$ in the interval $[a, b]$,*

3° *$f(x)$ has p zeros in the interval $(a, +\infty)$ which are not zero points of $f'(x)$ $(p < \infty)$,*

4° *$f(a) \neq 0$,*

5° *there is $c > a$ such that for every n $f_n(x) \neq 0$ for $x \geqslant c$, then the number of zeros of every function $f_n(x)$ for a sufficiently large n is equal to p in the interval $(a, +\infty)$.*

Proof. From the assumption 3° $f(x)$ has in the interval $(a, +\infty)$ p zero points; $p < \infty$. Then there exists $b < \infty$ such that in the interval (a, b) there are all zero points of the function $f(x)$. Hence $f(x) \neq 0$ for $x \geqslant b$. If we denote by $d := \max\{b, c\}$ then $f(x) \neq 0$ for $x \geqslant d$ and $f(x)$ has p zero points in the interval (a, d). Since in the interval (a, d) the sequences $f_n(x)$ and $f_n'(x)$ tend uniformly adequately to $f(x)$ and $f'(x)$, and $f(a) \neq 0$ and $(d) \neq 0$, then all assumptions of Lemma 4 from [1] hold. Hence there exist such $n_0 \in N$, fhat for every $n > n_0$ $f_n(x)$ has exactly p zero points in the interval (a, d) and because $t_n(x) \neq 0$ for $n \in N$, $x \geqslant d$, so for a sufficiently large n $f_n(x)$ has p zero points in the interval $(a, +\infty)$. The proof is complete.

LEMMA 5. *If:*

1° *$u_1(x), u_2(x), \ldots$ means a sequence of eigenfunctions such that it satisfies the conditions of* Assumption Z,

2° *$x_\gamma = \max\{x_k, x_s\}$, where x_k, x_s are the last zero points suitable of the functions $u_k(x)$, $u_s(x)$, then there exists M — constant such that*

$$\left|\frac{u_k(x)}{u_s(x)}\right| \leqslant M \quad \text{for } k,\ s = 1, 2, \ldots, n, \quad \text{and for } x > x_\gamma.$$

Proof. From Assumption Z $u_k(x)$ are continuous functions for $k = 1, 2, \ldots$ So contradicting the proposition we receive

$$\lim_{n\to\infty} \frac{u_k(x)}{u_s(x)} = \infty .$$

Hence

$$\int_{x_\gamma}^{+\infty} |u_k(x)|^2 dx = \int_{x_\gamma}^{+\infty} \left|\frac{u_k(x)}{u_s(x)}\right|^2 |u_s(x)|^2 dx$$

is divergent, which contradicts the statement that $u_k(x)$, $u_s(x)$ are of the class $\mathscr{L}_2$ in $[a, +\infty)$, so

$$\left|\frac{u_k(x)}{u_s(x)}\right| \leqslant M \qquad \text{for } x > x_\gamma$$

which was to be shown.

LEMMA 6. *If the function $f(x)$ is defined by formula (7) then there exists x_γ such that $f(x) \neq 0$ for $x > x_\gamma$.*

Proof. Each eigenfunction has a limited number of zero points. Let all zero points of the function $u_m(x), \ldots, u_n(x)$; $1 \leqslant m \leqslant n$, exist in the interval $[a, x_p]$. Hence

$$u_k(x) \neq 0 \qquad \text{for } k = m, \ldots, n \qquad \text{and for } x > x_p .$$

Let us denote by

(9) $\quad A := \{x;\ x > x_p,\ f(x) > 0 \quad$ and $f(x)$ each the local maximum at the point $x\}$

and by

$$f_1(x) = \lambda_m c_m u_m(x) + \ldots + \lambda_n c_n u_n(x) . \tag{10}$$

Let us observe that for $x \in A$

$$\frac{f_1(x)}{f(x)}$$

is bounded.

Really

$$\left|\frac{f_1(x)}{f(x)}\right| = \left|\frac{\lambda_m c_m u_m(x) + \ldots + \lambda_n c_n u_n(x)}{f(x)}\right| \leqslant \lambda_m \left|\frac{c_m u_m(x)}{f(x)}\right| + \ldots + \lambda_n \left|\frac{c_n u_n(x)}{f(x)}\right|$$

applying Lemma 5 we get that each quotient

$$\frac{c_k u_k(x)}{f(x)} \quad \text{is bounded for } x \in A \quad \text{and } k = m, \ldots, n .$$

Hence we get that there exists L such that

(11)
$$\left|\frac{f_1(x)}{f(x)}\right| \leqslant L \quad \text{for } x \in A .$$

$f(x)$ as a combination of eigenfunctions is the solution of the equation

(12)
$$p(x)f'(x) = q(x)f(x) - \varrho(x)[\lambda_m c_m u_m(x) + \ldots + \lambda_n c_n u_n(x)] .$$

Using the notation (10) and transforming (12) we receive

(13)
$$p(x)f''(x) = -p'(x)f'(x) + f(x)\left[q(x) - \varrho(x)\frac{f_1(x)}{f(x)}\right] \quad \text{for } x \in A .$$

Because $\lim_{x\to+\infty} q(x) = \infty$ and (11) for $x \in A$, then there exists a point x_λ such that

(14)
$$q(x) > \varrho(x)\frac{f_1(x)}{f(x)} \quad \text{for } x \in A \cap [x_\lambda, \infty) .$$

Let $x_0 \in A \cap [x_\lambda, \infty)$.
Let us calculate the value of the notation (13) in the point x_0

(15)
$$p(x_0)f''(x_0) = f(x_0)\left[q(x_0) - \varrho(x_0)\frac{f_1(x_0)}{f(x_0)}\right].$$

But (14) and the fact that $f(x_0) > 0$, shows that the right side of the formula (15) is positive, from the assumption $p(x_0) > 0$ so $f''(x_0) > 0$. Hence for $x > x_\lambda$ the function $f(x)$ cannot attain a positive maximum. Analogically we show that for $x > x_\lambda$, the function cannot attain a negative minimum. From this we have $f(x) \neq 0$ for $x \geqslant x_\lambda$.

§ 3. **Proof of the theorem.** It is easy to see that without loss of generality it can be assumed $c_m \neq 0$ and $c_n \neq 0$. For $n = 1$ the theorem is evident. Let us suppose that for $n > 1$ the function defined by formula (7) has p zero points in the interval (a, ∞). From the definition of the function $f(x)$ it follows that it satisfies the conditions of Lemma 3. Applying this lemma we get

(16)
$$h(x) = L[f(x)] = -\lambda_n \varrho(x)\left[\frac{\lambda_m}{\lambda_n} c_m u_m(x) + \ldots + c_n u_n(x)\right],$$

where $h(x)$ has at least p zero points in the interval $(a, +\infty)$. As $\lambda_n \neq 0$ (for $n > 1$) and $\varrho(x) \neq 0$ in $(a, +\infty)$ therefore the function

(17)
$$f_1(x) = \frac{\lambda_m}{\lambda_n} c_m u_m(x) + \ldots + c_n u_n(x)$$

has at least p zero points in the interval $(a, +\infty)$ and satisfies Lemma 6. But $f_1(x)$ has the same form as $f(x)$ and satisfies the assumptions of Lemma 3. Again applying Lemma 3 to $f_1(x)$ we obtain the function

$$f_2(x) = \left(\frac{\lambda_m}{\lambda_n}\right)^2 c_m u_m(x) + \ldots + c_n u_n(x) . \tag{18}$$

Proceeding thus we further obtain an infinite sequence of functions

$$f(x),\ f_1(x),\ f_2(x),\ \ldots, \tag{19}$$

for which the number of zeros does not decrease with the increase in the index and satisfies the assumptions of Lemma 6. Now

$$f_k(x) = \left(\frac{\lambda_m}{\lambda_n}\right)^k c_m u_m(x) + \ldots + c_n u_n(x) . \tag{20}$$

From (5) it follows that $0 < \frac{\lambda_s}{\lambda_n} < 1$ $(s = m, m+1, \ldots, n-1)$ and the functions $u_s(x)$ and $u_s'(x)$ $(s = m, m+1, \ldots, n-1)$, are bounded and square integrable in $(a, +\infty)$. Hence the sequence (19) and its limit $c_n u_n(x)$, for $k \to +\infty$, satisfy the assumptions of Lemma 4. Considering that the function $c_n u_n(x)$ has $n-1$ zeros in the interval $(a, +\infty)$ and therefore applying Lemma 4 to the sequence (19) and to any closed interval $[A, B]$ included in $(a, +\infty)$ for which zero points of the function $u_n(x)$ are interior points, we obtain the inequality

$$p \leqslant n-1 . \tag{21}$$

The proof of inequality

$$p \geqslant m-1 \tag{22}$$

is to be found in [1]. But the inequalities (21) and (22) are the actual thesis of Theorem 1.

References

[1] J. Bochenek, *On a certain question for the linear combinations of the eigenfunctions in the Sturm-Liouville problem*, Zeszyty Naukowe UJ, Prace Matematyczne 7 (1962), 43–47.

[2] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Oxford 1946.

INSTYTUT MATEMATYKI
POLITECHNIKA KRAKOWSKA
KRAKÓW (POLAND)