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Some conditions for the commutativity of differential rings

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Introduction. The purpose of the present paper is to give some conditions for the commutativity of differential rings. These conditions are of the type:

(α) $xx' = x'x$ for every x

or

(β) $x'-x$ is central for each x

or some generalization of (β),

(γ) for some natural number $k \geqslant 1$ $x^{(k)}-x$ is central for every x. It is not difficult to show that (β) is stronger than (α). We shall show that (α) is sufficient for the commutativity of such rings as have no nonzero differential nil-ideals and have all constants in the center. Under some additional assumptions, the condition (γ) is also sufficient for the commutativity of differential rings. We shall also show that a differential domain (i.e. a differential ring without proper zero divisors), which contains a nonzero constant, is commutative if and only if (α) is satisfied.

1. Definitions and notations. Let A be a ring. By a *derivation* of A we mean any additive map $D\colon A \to A$ satisfying

$$D(ab) = D(a)b + aD(b)$$

for every $a, b \in A$. A *differential ring* is a ring A with a derivation $'\colon A \to A$. If $x \in A$, then we define $x^{(0)} = x$, $x^{(n+1)} = (x^{(n)})'$ for any natural number $n \geqslant 0$. By A' we mean the smallest differential subring of A containing the set $\{x'\colon x \in A\}$. In the sequel we put

$$C(A) = \{x \in A\colon x' = 0\},$$

i.e. the ring of the constants of A and

$$Z(A) = \{x \in A\colon xy = yx \quad \text{for every } y \in A\},$$

i.e. the center of A. By an ideal of A we mean any two-sided ideal of A. An ideal I is said to be differential if: $x \in I \Rightarrow x' \in I$. We call a differential ideal I a d-nil-ideal if every element of I is nilpotent. By $N(A)$ we denote the largest of the d-nil-ideals of A (such a d-nil-ideal exists, because $\bigcup \{I\colon I \text{ is a } d\text{-nil-ideal}\}$ is a d-nil-ideal). It is clear that $N(A) = (0)$ if and only if A has no nonzero d-nil-ideals. We also have that $A/N(A)$ is a differential ring without nonzero d-nil-ideals, i.e. $N(A/N(A)) = (0)$. A differential ring without proper zero divisors is called the differential domain.

2. LEMMA 1. *Let A be a differential ring such that $N(A) = (0)$. If $a \in Z(A)$ fulfils $a^{(k)}a^{(m)} = 0$ for each $k, m \geq 0$, then $a = 0$.*

Proof. The set $aA + a'A + a^{(2)}A + \dots$ is a d-nil-ideal of A. Thus $aA = (0)$. Since the set $\{y \in A\colon\ yA = (0)\}$ is also a d-nil-ideal of A, we have $a = 0$.

LEMMA 2. *Let A be a differential ring satisfying*

$$zz' = z'z \tag{1}$$

for every $z \in A$. Then we have

(a) $x'y - yx' = xy' - y'x$,

(b) $x(x'y - yx') = (x'y - yx')x$,

(c) $y(xy - yx)(xy - yx)' = 0$

for every $x, y \in A$.

Proof. If we put $z = x + y$ in (1) then we obtain (a). Now let us put in (a) $a = xy$, $b = x$ first and $a = yx$, $b = x$ later. If we add these equalities, we obtain (b). Using (a) and (b) we have

$$x(xy' - y'x) = (xy' - y'x)x\,. \tag{2}$$

If we put $x = ba$, $y = b$ in (2), we obtain

$$b(ab - ba)(ab' - b'a) = 0\,.$$

From (a) we have

$$b(ab - ba)(a'b - ba') = 0\,.$$

If we add the last equalities, we obtain (c).

LEMMA 3. *Suppose that A is a differential ring without nonzero d-nil-ideals such that $zz' = z'z$ for every $z \in A$. If $C(A) \subset Z(A)$, then*

$$(xy_1 - y_1x)(xy_2 - y_2x) = 0$$

for any $x, y_1, y_2 \in A$.

Proof. From (b) in Lemma 2 we have

$$x'(xy - yx) = (xy - yx)x' \tag{3}$$

for every $x, y \in A$. Using (a), (b) of Lemma 2 and (3) we obtain

$$c = x(xy - yx) - (xy - yx)x \in C(A)\,.$$

If we put yx in the place of x we have $cx \in C(A)$. Since $C(A) \subset Z(A)$, so

$$x(xy - yx) = (xy - yx)x \tag{4}$$

for each $x, y \in A$. If we put $x = a$, $y = b+c$ in (4) we obtain

$$(ab-ba)c+(ac-ca)b = c(ab-ba)+b(ac-ca)\,. \tag{5}$$

Now let us put $a = x$, $b = xy$, $c = z$ in (5). Using again (4) and (5) we finish the proof of Lemma 3.

Next we give a definition.

Definition 1. Let A be a differential ring. A is called a *d-prime ring* if and only if the following implication holds true:

$$a^{(k)}Ab^{(m)} = (0), \quad a, b \in A, \quad k, m \geqslant 0 \Rightarrow a = 0 \quad \text{or } b = 0\,.$$

Remark 1. A differential ring is a d-prime ring if and only if the following implication holds true:

$IJ = (0)$, I, J are differential ideals of $A \Rightarrow I = (0)$ or $J = (0)$.

Proof. Implication $\Rightarrow$ is trivial. In order to prove the inverse implication we first observe that

$$xA = (0) \quad \text{or } Ax = (0) \Rightarrow x = 0\,. \tag{6}$$

Indeed, the differential ideals $I = \{y\colon yA = (0)\}$ or $I = \{y\colon Ay = (0)\}$ are such that $I^2 = (0)$. Thus $I = (0)$. This proves (6).

Now let $a^{(k)}Ab^{(m)} = (0)$ for some $a, b \neq 0$. The differential ideals $I = AaA+Aa'A+\ldots$ and $J = AbA+Ab'A+\ldots$ fulfils $IJ = (0)$. Hence $I = (0)$ or $J = (0)$. Let for example $I = (0)$. Thus $AaA = (0)$. Using (6) we obtain $a = 0$. This is a contradiction. The equivalence is proved.

If A and B are two differential rings, then a homomorphism $h\colon A \to B$ is said to be differential if $h(x') = (h(x))'$ for every $x \in A$.

PROPOSITION. *Let us assume that A is a differential ring such that $N(A) = (0)$. Then there exists a family $(I_\iota)_{\iota \in J}$ of differential ideals of A, such that*

1° *for every $\iota \in J$ the quotient ring $A_\iota = A/I_\iota$ is a d-prime ring and*

2° *A is differentially isomorphic with some differential subring of the direct product $\prod_{\iota \in J} A_\iota$.*

Proof. We take an arbitrarily fixed non-nilpotent element $a \in A$. By Kuratowski–Zorn's lemma there exists a differential ideal $I_a \subset A$, maximal with respect to the exclusion of the set $\{a^n\colon n \geqslant 1\}$. It is clear (from Remark 1) that A/I_a is a d-prime ring. Let us denote by J all the set of nonnilpotent elements of A. Since $a \notin I_a$ for every $a \in J$, $\bigcap \{I_x\colon x \in J\}$ is a d-nil-ideal of A. Thus from the assumptions of Proposition $\bigcap \{I_x\colon x \in J\} = (0)$. Hence the mapping

$$A \ni y \to (y_x)_{x \in J} \in \prod_{x \in J} A/I_x$$

is a differential monomorphism. The proof of Proposition is completed.

LEMMA 4. *Let A be a differential ring such that $N(A) = (0)$. Suppose that $x'(xy-yx) = 0$ for every $x, y \in A$. If $z \in A$ is an element satisfying $z^{(k)}z^{(m)} = 0$ for each $k \geqslant 1$, $m \geqslant 0$, then $z \in C(A)$.*

Proof. In order to prove the Lemma we first show that it holds true for d-prime rings. Let us assume that A is a d-prime ring. We show by induction (for k) that $z^{(k)}Az^{(m)} = (0)$ for every $k \geqslant 1$, $m \geqslant 0$. From Definition 1 we obtain $z \in C(A)$.

Now let A be a differential ring having no nonzero d-nil-ideals. In virtue of Proposition there exists a family $(I_\iota)_{\iota \in J}$ of the ideals I_ι and differential subring B of $\prod_{\iota \in J} A/I_\iota$ such that A is differentially isomorphic with B. It is clear that it is sufficient to prove the lemma for B. Let $z = (z_\iota)_{\iota \in J}$ be an element of B such that $z^{(k)}z^{(m)} = 0$ for $k \geqslant 1$, $m \geqslant 0$. Since A/I_ι is a d-prime ring and fulfils the assumptions of Lemma 4, we obtain $z_\iota \in C(A/I_\iota)$ for every $\iota \in J$. Hence $z \in C(B)$ This completes the proof.

3. **THEOREM** 1. *Assume that R is a differential ring without nonzero d-nil-ideals such that $zz' = z'z$ for every $z \in R$. If $C(R) \subset Z(R)$, then R is a commutative ring.*

Proof. From Lemma 2(b) we have

$$x'(xy-yx) = (xy-yx)x' \tag{7}$$

for every $x, y \in R$. If we put $x = b+c$, $y = a$ we obtain

$$(ca-ac)b'-b'(ca-ac) = c'(ba-ab)-(ba-ab)c' . \tag{8}$$

Now let us put $a = x$, $b = xy$, $c = z$ in (8). We have

$$x'(zx-xz)y = x'y(zx-xz) . \tag{9}$$

It follows from Lemma 3 that $x'(zx-xz) \in Z(R)$ for each $x, z \in R$. It is easy to see that $a = x'(zx-xz)$ fulfils the assumptions of Lemma 1. Thus $x'(zx-xz) = 0$ for every $x, z \in R$. Hence the assumptions of Lemma 4 are satisfied. Because $(xy-yx)^{(k)}(xy-yx)^{(m)} = 0$ for every $x, y \in R$ and $k, m \geqslant 0$, we have $xy-yx \in C(R) \subset Z(R)$. From Lemma 1 we obtain that R is a commutative ring.

As an important corollary we have:

COROLLARY 1. *If a differential ring R having no nonzero d-nil-ideals is such that $x'-x \in Z(R)$ for every $x \in R$, then R is a commutative ring.*

Remark 2. If a differential ring A fulfils $zz' = z'z$ for every $z \in A$, then $C(A') \subset Z(A')$. This is an immediate consequence of Lemma 2(a).

COROLLARY 2. *We assume that R is a differential ring such that $xx' = x'x$ for any $x \in R$. If R' is such that $N(R') = 0$, then R' is a commutative ring.*

Proof. We take $A = R'$ and apply Theorem 1. The assumptions of Theorem 1 are fulfilled (see Remark 2). Thus R' is a commutative ring.

COROLLARY 3. *Suppose that R is a differential ring such that $xx' = x'x$ for every $x \in R$. Denote by J the differential ideal in R' generated by the commutators of R' (i.e. by the set $\{xy-yx\colon x, y \in R\}$). Then $J \subset N(R')$.*

Proof. It is clear that R'/I is a ring that $N(R'/N(R')) = (0)$ and that $R'/N(R') = (R/N(R'))'$. By Corollary 2 $R'/N(R')$ is commutative. Thus $xy-yx \in N(R')$ for every $x, y \in R$. Hence $J \subset N(R')$.

THEOREM 2. *If R is a differential domain such that $C(R) \neq R$ and $xx' = x'x$ for every $x \in R$, then R is a commutative ring.*

Proof. Because R is a ring without proper zero divisors, by Lemma 2(c) we obtain that $xy-yx \in C(R)$ for every $x, y \in R$. Now we show the following implication:

$$xy \neq yx \Rightarrow x \in C(R)\,. \tag{10}$$

Indeed, $x(xy-yx) = x(xy)-(xy)x \in C(R)$. Hence $x \in C(R)$, since $xy-yx \neq 0$. Thus (10) holds true. Now we take $a \in R$ such that $a' \neq 0$. Assume that $bc \neq cb$ for some $b, c \in R$. In virtue of (10) we have $b \in C(R)$ and $ac = ca$. Thus $ab \notin C(R)$ and $(ab)c = c(ab)$ by the implication (10), again. Hence $a(bc-cb) = 0$. This is a contradiction. The proof of Theorem 2 is completed.

4. Now we give a generalization of Corollary 1.

LEMMA 5. *Let Q be a ring, G an additive subgroup of Q and $\{x_1, \ldots, x_n\} \subset Q$. If matrix $K = (k_{ij})_{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n}$, k_{ij} are entires, $\det K \neq 0$ is such that $\sum_{j=1}^{n} k_{ij}x_j \in G$ for every $i = 1, \ldots, n$, then there exists a positive entire N such that $Nx_i \in G$ for each $i = 1, \ldots, n$.*

Proof. The proof of the lemma is trivial, we may take $N = |\det K|$.

THEOREM 3. *Assume R is a differential ring and that $N(R) = (0)$. Let $k \geqslant 1$ be a fixed natural number. Suppose that*

$$a^{(k)} - a \in Z(R) \tag{11}$$

for every $a \in R$. Then for every $x, y \in R$ there exists a positive entire N such that $N(xy-yx) = 0$.

Proof. By assumption (11) we have:

$$(xy-yx)^{(k)} - (xy-yx) = \binom{k}{0}(xy-yx) + \binom{k}{1}(x^{(k-1)}y' - y'x^{(k-1)}) + \ldots$$
$$\ldots + \binom{k}{k-1}(x'y^{(k-1)} - y^{(k-1)}x')$$

$$(x^{(k-1)}y' - y'x^{(k-1)})^{(k)} - (x^{(k-1)}y' - y'x^{(k-1)}) = \binom{k}{0}(x^{(k-1)}y' - y'x^{(k-1)}) +$$
$$+\binom{k}{1}(x^{(k-2)}y^{(2)} - y^{(2)}x^{(k-2)}) + \ldots + \binom{k}{k-1}(xy-yx)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$(x'y^{(k-1)} - y^{(k-1)}x')^{(k)} - (x'y^{(k-1)} - y^{(k-1)}x') = \binom{k}{0}(x'y^{(k-1)} - y^{(k-1)}x') +$$
$$+\binom{k}{1}(xy-yx) + \ldots + \binom{k}{k-1}(x^{(2)}y^{(k-2)} - y^{(k-2)}x^{(2)})$$

for every $x, y \in R$. We denote the matrix

$$\begin{bmatrix} \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k-1} \\ \binom{k}{k-1} & \binom{k}{0} & \cdots & \binom{k}{k-2} \\ \vdots & \vdots & & \vdots \\ \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{0} \end{bmatrix}$$

by M. It is known (see for instance [4]) that $\det M = \varphi(\varepsilon_1) \ldots \varphi(\varepsilon_k)$, where $\varepsilon_1, \ldots, \varepsilon_k \in C$ are all the k-roots of 1 and $\varphi(x) = \binom{k}{0} + \binom{k}{1}x + \ldots + \binom{k}{k-1}x^{k-1}$. We show that $\det M \neq 0$. Assume on the contrary that $\varphi(\varepsilon_i) = 0$ for some $i \in \{1, \ldots, k\}$. Then the polynomial $\varphi(x)$ is a divisor of the polynomial $x^k - 1$. This is a contradiction. Thus $\det M \neq 0$. If follows from Lemma 5 (for $Q = R, G = Z(R)$) that $N(xy-yx) \in Z(R)$ for some $N \geqslant 1$. By Lemma 1 we have $N(xy-yx) = 0$ (it is easy to see that $a = N(xy-yx)$ fulfils the assumptions of this lemma). The proof of Theorem 3 is completed.

As a simple consequence of Theorem 3 we have the following:

COROLLARY 4. *Assume that R is a differential ring satisfying the implication*

$$(12) \qquad ma = 0, \quad m \text{ a entire}, \quad a \in R \Rightarrow m = 0$$

and such that $N(R) = (0)$. If for some $k \geqslant 1$, $a^{(k)} - a \in Z(R)$ for every $a \in R$, then R is a commutative ring.

COROLLARY 5. *Let R be a differential ring satisfying implication* (12). *Assume that there exists $k \geqslant 1$ such that $a^{(k)} - a \in Z(R)$ for every $a \in R$. Then the differential ideal generated by commutators of R is contained in $N(R)$.*

5. Example. The next example shows that the assertions of Corollary 1 (and Theorem 1) are false if we omit the assumption that the ring considered has no nonzero d-nil-ideals.

Let F be any fixed field and let V be a vector space over F (we assume that $\dim_F V \geqslant 3$). Let $B \subset V$ be a basis of V and x_1, y_1, z_1 be three arbitrarily fixed different vectors of the basis B. For $x, y \in B$ we put

$$xy = \begin{cases} z_1 & \text{if } x = x_1 \text{ and } y = y_1, \\ 0 & \text{if } x \neq x_1 \text{ or } \ y \neq y_1. \end{cases}$$

This operation on B may be extended by linearity on V. In this way we obtain a noncommutative algebra over F. Now we define for $x \in B$

$$x' = \begin{cases} x & \text{if } x \neq z_1, \\ 2x & \text{if } x = z_1. \end{cases}$$

We obtain a derivation on V (V is a ring, in particular) extending by linearity this mapping on V. We obtain $C(V) = (0)$ for F of characteristic $\neq 2$ and otherwise $C(V) = \{\alpha z_1 : \alpha \in F\}$.

We obtain $C(V) \subset Z(V)$ and $x' - x \in Z(V)$ for $x \in V$, in both cases. The differential ring V fulfils all assumptions of Corollary 1 except $N(V) = (0)$ (because $\{\alpha z_1 : \alpha \in F\}$ is a d-nil. ideal of V) and it is a noncommutative ring.

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