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On some inequalities for polynomials

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Let E be a Borel subset of the space C^N of N complex variables and let μ be a positive measure defined on E. We say that the pair (E, μ) satisfies the *condition* $\mathscr{L}^*$ if for every $b>1$ and for every family $\mathscr{F}$ of polynomials of N complex variables satisfying

$$\sup_{f\in\mathscr{F}} |f(x)| < +\infty \quad \mu\text{-almost everywhere on } E \tag{1}$$

there exists an open neighbourhood G of E and a positive constant M such that

$$\|f\|_G \leqslant Mb^{s(f)}, \quad f\in\mathscr{F}, \tag{2}$$

where $s(f) := \deg f$, $\|f\|_G := \sup_{x\in G} |f(x)|$.

Remark 1. If (E, μ) satisfies $\mathscr{L}^*$ and ν is a positive measure on E such that $\nu(e) = 0 \Rightarrow \mu(e) = 0$ for all Borel sets $e\subset E$, then (E, ν) satisfies $\mathscr{L}^*$. For instance if w is a real measurable function positive μ-almost everywhere on E and $d\nu := w d\mu$, then (E, ν) satisfies $\mathscr{L}^*$.

PROPOSITION 1. *Given any pair (E, μ) (with E bounded) the following conditions are equivalent*:

(a) (E, ν) *satisfies* $\mathscr{L}^*$;

(a_0) *For every $b>1$ there exists an open neighbourhood G of $\bar{E}$ such that for every family $\mathscr{F}$ satisfying* (1) *one can find a positive constant M such that* (2) *is true*;

(b) *For every $b>1$ there exists an open neighbourhood G of $\bar{E}$ such that for every sequence of polynomials $\{f_k\}$ with $\sup_k |f_k(x)| < +\infty$ μ-almost everywhere on E one can find a positive constant M such that $\|f_k\|_G \leqslant Mb^{s(f_k)}$, $k\geqslant 1$*;

(c) *For every $b>1$ there exists an open neighbourhood G of $\bar{E}$ such that for every sequence of polynomials $\{f_k\}$ with $s(f_k)\leqslant k$ and $\sup_k |f_k(x)| < +\infty$ μ-almost everywhere on E one can find a positive constant M such that $\|f_k\|_G \leqslant Mb^k$, $k\geqslant 1$.*

Proof. First we shall prove that (a)$\Rightarrow$(a_0).

Suppose (a_0) to be not true. Then there exists $b>1$ such that for every $k\geqslant 1$ one can find a sequence of polynomials $\{f_{kn}\}_{n\geqslant 1}$ satisfying

$$\sup_x |f_{kn}(x)| < +\infty \quad \mu\text{-almost everywhere on } E$$

and

(i) $$\|f_{kn}\|_{G_k} > nb^{s(f_{kn})}, \quad n, k \geq 1,$$

where $G_k := \left\{x \in C^N :\ \mathrm{dist}(x, E) < \frac{1}{k}\right\}$.

Put $b_1 = \sqrt{b}$. By (a) for every $k \geq 1$ one can find an open set $\Omega_k \supset \bar{E}$ and a positive constant M_k such that

$$\|f_{kn}\|_{\Omega_k} \leq M_k b_1^{s(f_{kn})}, \quad k, n \geq 1.$$

In particular, $|f_{kn}(x)/M_k b_1^{s(f_{kn})}| \leq 1$ for all $x \in E$ and $n, k \geq 1$. Hence, again by (a), one can find an open set $G \supset \bar{E}$ and a constant $M > 0$ such that

(ii) $$\|f_{kn}\|_G \leq MM_k b^{s(f_{kn})}, \quad k, n \geq 1.$$

Take k_1 so large that $G_{k_1} \subset G$. Then by (i) and (ii) we get

$$n \leq MM_{k_1}, \quad n \geq 1.$$

This contradiction proves that (a_0) follows from (a).

It is obvious that (a_0)$\Rightarrow$(b)$\Rightarrow$(c).

Assume that (c) is true and (a) is not. Then there exist $b > 1$ and a sequence of polynomials $\{f_k\}$ such that

(3) $$\|f_k\|_{G_k} > b^{2k+s(f_k)}, \quad k \geq 1, \quad \text{where } G_k := \{x \in C^N :\ \mathrm{dist}(x, \bar{E}) < 1/k\}.$$

Consider two cases.

1° There exists m such that $s(f_k) \leq m$ for all k. Then $s(f_k) \leq k$ for $k \geq m$. Hence by (c) $\exists_{G \supset E} \exists_{M > 0}$ such that $\|f_k\|_G < Mb^k$, $k \geq m$. This however contradicts (3).

2° If $\{s(f_k)\}$ is not bounded we may assume that $s(f_k) < s(f_{k+1})$. Then by (c) we can find $G \supset \bar{E}$ and $M > 0$ such that $\|f_k\|_G < Mb^{s(f_k)}$. This again contradicts (3).

Let us recall some examples of pairs (E, μ) satisfying $\mathscr{L}^*$.

Ex. 1. (F. Leja [3]). $N = 1$, $E = [a, b]$ — a compact interval of $\boldsymbol{R}$. $\mu = \lambda$ — the Lebesgue measure on $\boldsymbol{R}$. More generally, E may be a rectifiable Jordan arc on C, μ — the length measure on E.

Now by Fubini's theorem we get the following

Ex. 2. $E = I^N \subset \boldsymbol{R}^N$, where $I = [0, 1]$; $\mu = \lambda_N$ — the N-dimensional Lebesgue measure in $\boldsymbol{R}^N$.

From Ex. 2 one immediately obtains

Ex. 3. $E = \varphi(I^N)$, where $\varphi \in GL(\boldsymbol{R}^N)$, $\mu = \lambda_N$.

Ex. 4. (Dudley and Randol [1], N.T.V. [5]). If $\varphi \in GL(\boldsymbol{R}^N)$ the set $Q = \varphi(I^N)$ is called an N-dimensional parallelepiped.

Let E be a bounded subset of $\boldsymbol{R}^N$ such that for every point $x \in \bar{E}$ there exists a parallelepiped Q_x such that $x \in Q_x$ and $Q_x \subset E \cup \{x\}$. (In particular as E we can take any bounded convex domain in $\boldsymbol{R}^N$, or any bounded domain in $\boldsymbol{R}^N$ with Lipschitz boundary.) Then the pair (E, λ_N) satisfies $\mathscr{L}^*$.

Ex. 5. (N.T.V. [4]). $N = 1$, E — non-effilé compact subset of C, μ — the harmonic measure on E.

Let $S = S(E)$ be a subset of $\bar{E}$ such that for every polynomial f there exists $x_0 \in S$ such that $\| f \|_E = |f(x_0)|$. We can now state the main result of this note.

THEOREM. *Let E be a Borel subset of C^N and let μ be a positive measure on E such that (E, μ) satisfies $\mathcal{L}^*$ and μ satisfies the following condition*

$$(+) \qquad \forall_{x_0 \in S} \forall_{r>0} \ \mu(B_r \cap E) > 0, \quad \text{where } B_r := \{|x - x_0| < r\} .$$

Then for every $p>0$ and for every $b>1$ there exists an open neighbourhood G of $\bar{E}$ and a positive constant M such that for every polynomial f of N complex variables

$$(4) \qquad \| f \|_G \leqslant M b^{s(f)} \| f \|_{\mu,p} ,$$

where $\| f \|_{\mu,p} := (\int_E |f(x)|^p d\mu)^{1/p}$.

Proof. First we shall prove that for every $p>0$ and for every $b>1$ there exists a constant $M>0$ such that for every polynomial f

$$(5) \qquad \| f \| := \| f \|_E \leqslant M b^{s(f)} \| f \|_{\mu,p} .$$

If (5) were not true we could find $p>0$, $b>1$ and a sequence of polynomials $\{f_k\}$ such that

$$(6) \qquad \| f_k \| > k b^{s(f_k)} \| f_k \|_{\mu,p} \quad \text{for all } k .$$

It follows that $\| f_k \| > 0$ and $0 < \| f_k \|_{\mu,p} < +\infty$, $k \geqslant 1$. Dividing both sides of (6) by $\| f_k \|$ we may assume that $\| f_k \| = 1$. We claim that the sequence $\{s(f_k)\}$ is not bounded. Indeed, if $s(f_k) \leqslant m$ $(k \geqslant 1)$ for some m, then without loss of generality we may assume that the sequence $\{f_k\}$ is uniformly convergent on E. Hence, since $\| f_k \| = 1$, we can find $x_0 \in S = S(E)$ and $r>0$ such that $|f_k(x)| > 1/2$ for $x \in B \cap E$, $k \geqslant k_0$, where $B = \{|x - x_0| < r\}$. Therefore

$$k^{-p} > \| f_k \|^p_{\mu,p} \geqslant \int_{B \cap E} |f_k(x)|^p d\mu \geqslant 2^{-p} \mu(B \cap E) > 0, \qquad k \geqslant 1 .$$

This contradiction proves that the sequence $\{s(f_k)\}$ is not bounded. We may assume that $s(f_k) < s(f_{k+1})$, $k \geqslant 1$. We claim that for every $q>1$ the sequence of polynomials $g_k := q^{-s(f_k)} f_k / \| f_k \|_{\mu,p}$ is bounded μ-almost everywhere on E. Indeed, following N.T.V. [5], put

$$E_{n,k} := \{x \in E\colon \ |g_k(x)| \geqslant n\}, \qquad E_n := \bigcup_{k=1}^{\infty} E_{n,k}$$

and observe that

$$\mu(E_n) \leqslant \sum_{k=1}^{\infty} n^{-p} q^{-ps(f_k)} \leqslant n^{-p} \frac{q^p}{q^{p-1}}, \qquad n \geqslant 1 ,$$

whence it follows that $\{g_k\}$ is bounded μ-almost everywhere on E. Now by the assumption that (E, μ) satisfies $\mathscr{L}^*$ we can find $G \supset \bar{E}$ and $M>0$ such that $\|g_k\|_G \leqslant Mq^{s(f_k)}$, $k \geqslant 1$. Hence

$$\|f_k\|_G \leqslant Mq^{2s(f_k)} \|f_k\|_{\mu,p}, \quad k \geqslant 1 . \tag{7}$$

Take $q = b^{1/2}$. Then (7) and (6) imply $k < M$ for all k. This contradiction proves that (5) is true. Now applying (5) and the assumption that (E, μ) satisfies $\mathscr{L}^*$, we get (4).

COROLLARY. *Put* $c_k = \sup\{\|f\|_E / \|f\|_{\mu,p} : f \in P_k\}$, *where* P_k *denotes the family of all polynomials of degree at most* k. (*By definition we put* $c_k := 0$, *when* $\|f\|_{\mu,p} = 0$ *for all* $f \in P_k$.)

Then $\limsup_{k\to\infty} c_k^{1/k} \leqslant 1$. *If* $\mu(E) < \infty$, *then* $\lim c_k^{1/k} = 1$.

Remark 2. Our Theorem implies as a special case Théorème 1 of [5].

Problem. Under what assumptions on E and μ have we $c_k \leqslant Mk^\beta$ for some constants $M>0$ and $\beta>0$? A partial answer is given in [2].

References

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