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Equivalence of two definitions of statistical solution of the heat equation

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1. Introduction. We compare two definitions of the statistical solution of a differential equation in the case of the heat equation. In both definitions the statistical solution is a family of measures $\{\mu_t\}$ satisfying certain conditions. The first definition given in Section 3 is based on [1]. This approach enables the effective formulae of the moment functions of measures μ_t to be found. The second definition given in Section 4 is a slight modification of the definition of Foiaş and Prodi (see [2]) given for the Navier-Stokes equations (cf. also [3] where the theory of Foiaş and Prodi has been adapted to the case of the heat equation). In this case we derive some equations characterising the moment functions. A proof of the equivalence of both definitions is given in Section 5.

2. First integrals of the heat equation. Let Ω be a bounded domain in R^3 with boundary of class C^2. We consider the following initial-boundary problem

$$(1)\qquad \begin{cases} \dfrac{\partial}{\partial t}u(t,u)-\Delta u(t,x)=f(t,x) & \text{for } t\in(0,T),\ x\in\Omega;\\ u(0,x)=u_0(x) & \text{for } x\in\Omega;\\ u(t,\cdot)\in H_0^2(\Omega) & \text{for } t\in(0,T), \end{cases}$$

where $T\in R$ or $T=\infty$, u_0 is given in $H_0^2(\Omega)$ and f is given in $C(0,T;H_0^2(\Omega))$. We assume that $u(\cdot,x)\in C^1((0,T))\cap C([0,T))$ and Δu is understood in the sense of the theory of distributions. We also consider the following problem

$$(2)\qquad \begin{cases} \dfrac{\partial}{\partial t}v(t,\xi)+|\xi|^2v(t,\xi)+\hat{f}(t,\xi) & \text{for } \xi\in R^3,\ t\in(0,T);\\ v(0,\xi)=\hat{u}_0(\xi) & \text{for } \xi\in R^3, \end{cases}$$

where $\hat{f}(t,\cdot)$ is the Fourier transform (¹) of $f(t,\cdot)$ and for $g\in H_0^2(\Omega)$ $\hat{g}$ denotes the Fourier transform of

$$G(x)=\begin{cases} g(x) & \text{for } x\in\Omega,\\ 0 & \text{for } x\in R^3\setminus\Omega. \end{cases}$$

(¹) We denote $\hat{F}(\xi)=\mathcal{F}_{x\to\xi}F(x)=(2\pi)^{-3}\int_{R^3}e^{ix\xi}F(x)\,dx,\ x\xi=\sum_{i=1}^{3}x_i\xi_i\cdot$

LEMMA. *If $g \in H_0^2(\Omega)$ then $\hat{g} \in L^1(R^3)$.*

Proof. $G \in L^2(R^3)$ and $D_i^k G \in L^2(R^3)$ hence G and $D_i^k G$ belong to $L^1(R^3)$ $(i = 1, 2, 3;\ k \leqslant 2)$. From the Plancherel theorem we get

$$\int_{R^3} |\hat{G}(\xi)|^2 d\xi < \infty$$

and for $k = 2$

$$\int_{R^3} \xi_i^4 |\hat{G}(\xi)|^2 d\xi < \infty \qquad (i = 1, 2, 3).$$

From these inequalities we clearly get

$$\int_{R^3} (1+|\xi|)^4 |\hat{G}(\xi)|^2 d\xi < \infty .$$

Finally applying the Schwarz inequality we have

$$\Big(\int_{R^3} |G(\xi)| d\xi\Big)^2 \leqslant \int_{R^3} [(1+|\xi|)^2 |\hat{G}(\xi)|]^2 d\xi \cdot \int_{R^3} (1+|\xi|)^{-4} d\xi < \infty .$$

This finishes the proof of the Lemma (cf. the proof of the Sobolev Lemma in [4]).

We denote

$$W = \{v(\xi)\colon \mathscr{F}^{-1}_{\xi \to x} v(\xi) \in H_0^2(\Omega)\}$$

which is, from the Lemma, a subset of $L^1(R^3)$. We consider W with the topology induced from $H_0^2(\Omega)$ by $\mathscr{F}_{x \to \xi}$.

Definition 1. A functional $G\colon I \times H_0^2(\Omega) \to C$ (I denotes any interval in R) such that $G(t, u(t, x)) =$ const for any $u(t, x)$ being the solution of (1), will be called *the first integral* of (1).

Definition 2. A functional $\varphi\colon I \times W \to C$ such that $\phi(t, v(t, \xi)) =$ const for any $v(t, \xi)$ being the solution of (2), will be called *the first integral* of (2).

There exists a 1–1 correspondence between the first integrals of (1) and (2). We denote by $\mathscr{T}_2$ the set of continuous functionals $\phi\colon L^1(R^3) \to C$ having a Fréchet–Volterra derivative $\dfrac{\delta\phi(v)}{\delta v(\xi)}$. $\mathscr{T}_1 = \{G\colon H_0^2(\Omega) \to C$ such that $G \circ \mathscr{F}^{-1}_{\xi \to x} = \phi|_W$ for some $\phi \in \mathscr{T}_2\}$. Any first integral of (2) belonging to $\mathscr{T}_2$ satisfies

$$\frac{\partial\phi(t, v)}{\partial t} - \int_{R^3} \frac{\delta\phi(t, v)}{\delta v(\xi)} \big(|\xi|^2 v(\xi) - \hat{f}(t, \xi)\big) d\xi = 0 . \tag{3}$$

We solve (3) with the initial condition

$$\phi(0, v) = \psi(v) .$$

Applying the method of characteristics (cf. [5], [6]) we get the unique solution of (3) given by the following formula

$$\phi(t, v) = \psi\Big(e^{|\xi|^2 t} v(\xi) - \int_0^t \hat{f}(\tau, \xi) e^{|\xi|^2 \tau} d\tau\Big) . \tag{4}$$

The domain of ψ is $L^1(R^3)$ thus $\phi(t, v)$ is well defined for $t \leqslant 0$. We extend $\hat{f}$ to $t<0$ and (4) will then have sense. We put

$$\bar{f}(t, \xi) = \begin{cases} \hat{f}(t, \xi), & \text{for } t \geqslant 0; \\ \hat{f}(-t, \xi) e^{-t|\xi|^2} - |\xi|^2 \int\limits_0^{-t} \hat{f}(\tau, \xi) e^{\tau|\xi|^2} d\tau, & \text{for } t<0. \end{cases}$$

Then (4) takes the form

$$\phi(t, v) = \psi\Big(e^{|\xi|^2 t} v(\xi) - \int\limits_2^t \bar{f}(\tau, \xi) e^{|\xi|^2 \tau} d\tau\Big)$$

for $t \leqslant 0$. Hence for $t \geqslant 0$ we have

$$\phi(-t, v) = \psi\Big(e^{-|\xi|^2 t} v(\xi) + \int\limits_0^t \bar{f}(-s, \xi) e^{-|\xi|^2 s} ds\Big).$$

But clearly

$$\int\limits_0^t \bar{f}(-s, \xi) e^{-|\xi|^2 s} ds = e^{-|\xi|^2 t} \int\limits_0^t \hat{f}(s, \xi) e^{|\xi|^2 s} ds$$

so that we finally get

$$\phi(-t, v) = \psi\Big(e^{-|\xi|^2 t} v(\xi) + e^{-|\xi|^2 t} \int\limits_0^t \hat{f}(s, \xi) e^{|\xi|^2 s} ds\Big) \tag{5}$$

for $t \geqslant 0$. Thus we have proved the following theorem

THEOREM 1. *The problem of seeking the first integral of* (1) (*resp.* (2)) *with initial functional* $H \in \mathcal{T}_1$ (*resp.* $\psi \in \mathcal{T}_2$) *has the unique solution in* $\mathcal{T}_1$ (*resp.* $\mathcal{T}_2$).

3. Statistical solutions and moment functions. Theorem 1 enables us to give the following definitions.

Definition 3. Let μ^0 be a probability on $H_0^2(\Omega)$ with the support contained in some compact set $K \subset H_0^2(\Omega)$. A family of probabilities $\{\mu_t\}$, $t \in [0, T)$ on $H_0^2(\Omega)$ with $\operatorname{supp} \mu_t \subset K$ (for each $t \in [0, T)$) will be called *the statistical solution* of (1) with the initial measure μ^0 if and only if

$$\int\limits_K H(u) d\mu_t(u) = \int\limits_K G_H(-t, u) d\mu^0(u) \tag{6}$$

holds for each $t \in [0, T)$ and $H \in \mathcal{T}_1$ where G_H is the unique first integral of (1) satisfying $G_H(0, u) = H(u)$.

Definition 4. Let $\tilde{\mu}^0$ be a probability on W with the support contained in some compact set $\tilde{K} \subset W$. A family of probabilities $\{\mu_t\}$ $t \in [0, T)$ on W with $\operatorname{supp} \tilde{\mu}_t \subset \tilde{K}$ ($t \in [0, T)$) will be called *the statistical solution* of (2) with the initial measure μ^0 if and only if

$$\int\limits_{\tilde{K}} \psi(v) d\tilde{\mu}_t(v) = \int \phi_\psi(-t, v) d\mu^0(v) \tag{6'}$$

for each $t \in [0, T)$ and $\psi \in \mathcal{T}_2$ where ϕ_ψ is the unique first integral of (2) satisfying $\phi_\psi(0, v) = \psi(v)$.

It can be seen that there is 1–1 correspondence between the statistical solutions of (1) and (2) given by

$$\tilde{\mu}_t(\omega) = \mu_t(\mathcal{F}^{-1}_{\xi\to x}(\omega)) \quad \text{for } \omega \subset W. \tag{7}$$

Functions

$$M_n(t, x_1, \ldots, x_n) = \int_K u(x_1) \ldots u(x_n) d\mu_t(u)$$

($n = 1, 2, \ldots$) will be called the moment functions of the measure μ_t. It may simply be proved that

$$M_n(t, x_1, \ldots, x_n) = \int_{R^3} \ldots \int_{R^3} e^{i(x_1\eta_1 + \ldots + x_n\eta_n)} \tilde{M}_n(t, \eta_1, \ldots, \eta_n) d\eta_1 \ldots d\eta_n \tag{8}$$

where $\tilde{M}_n$ are the moment functions of the measure $\tilde{\mu}_t$.

THEOREM 2. *Let* $\{\mu_t\}$, $t \in [0, T)$ *be any statistical solution of* (1) *with the initial measure* $\mu^0 l$ *and* $\{\tilde{\mu}_t\}$, $t \in [0, T)$ *the corresponding* (see (7)) *statistical solution of* (2) *with the initia, measure*

$$\tilde{\mu}^0(\omega) = \mu^0(\mathcal{F}^{-1}_{\xi\to x}(\omega)), \qquad \omega \subset W.$$

Then we have the following formulae

$$\begin{aligned} M_n(t, x_1, \ldots, x_n) &\\ = \int e^{i\langle x,\eta\rangle} e^{-t\sum\limits_{i=1}^{n} |\eta_i|^2} & \sum_{\alpha^s} \{\tilde{M}^0_n(\eta_{\alpha_1}, \ldots, \eta_{\alpha_s}) \prod_{i=1}^{n-s} \int_0^t \hat{f}(\tau, \eta_{\beta_i}) e^{|\eta_{\beta_i}|^2\tau} d\tau\} d\eta_1 \ldots d\eta_n, \end{aligned} \tag{9}$$

where $\tilde{M}^0_n$ *are the moment functions of* $\tilde{\mu}^0$ (cf. [1], p. 376). *The sum is taken over all* $\alpha^s = \{\alpha_1, \ldots, \alpha_s\}$ *subsets of* $\{1, \ldots, n\}$ *and we denote* $\{1, \ldots, n\} \setminus \alpha^s = \{\beta_1, \ldots, \beta_{n-s}\}$, $\langle x, \eta\rangle = \sum\limits_{i=1}^{n} x_i\eta_i$.

Proof. We take ψ of the form

$$\psi(v) = \int e^{i\langle x,\eta\rangle} v(\eta_1) \ldots v(\eta_n) d\eta_1 \ldots d\eta_n .$$

From (5) and (6′) we get

$$\begin{aligned} \int_{\tilde{K}} \psi(v) d\tilde{\mu}_t &= \int_{\tilde{K}} \psi(ve^{-|\xi|^2 t} + e^{-|\xi|^2 t} \int_0^t \hat{f}(s, \eta_i) e^{|\eta_i|^2 s} ds) d\tilde{\mu}^0(v) \\ &= \int_{\tilde{K}} \int e^{i\langle x,\eta\rangle} \prod_{i=1}^{n} [v(\eta_i) e^{-|\eta_i|^2 t} + e^{-|\eta_i|^2 t} \int_0^t \hat{f}(s, \eta_i) e^{|\eta_i|^2 s} ds] d\tilde{\mu}^0(v) d\eta_1 \ldots d\eta_n \\ &= \int e^{i\langle x,\eta\rangle} e^{-t\sum\limits_{i=1}^{n} |\eta_i|^2} \sum_{\alpha^3} [\int_{\tilde{K}} v(\eta_{\alpha_1}) \ldots v(\eta_{\alpha_3}) d\tilde{\mu}^0(v) \prod_{i=1}^{n-s} \int_0^t \hat{f}(s, \eta_{\beta_i}) e^{|\eta_{\beta_i}|^2 s} ds] d\eta_1 \ldots d\eta_n . \end{aligned}$$

Hence applying (8) we obtain (9).

4. The Foiaş-Prodi definition and the equations on moment functions. In this Section we restrict our attention to families of Borel probabilities $\{\mu_t\}$ on $H_0^2(\Omega)$ satisfying the following assumptions

1° $\operatorname{supp}\mu_t \subset K$ (compact set in $H_0^2(\Omega)$) for $t \in [0, T)$,

2° all moment functions are of class $C^1((0, T)) \cap C([0, T))$ with respect to t, and of class C^2 with respect to each spatial variable,

3° $\sup \left| \frac{\partial}{\partial t} M_n(t, x_1, \ldots, x_n) \right| \leqslant c_1$.

Definition 5. We take any probability μ^0 on $H_0^2(\Omega)$ with $\operatorname{supp}\mu^0 \subset K$. A family $\{\mu_t\}$, $t \in [0, T)$ of Borel probabilities on $H_0^2(\Omega)$ satisfying 1°–3° will be called *the statistical solution of* (1) (in the sense of the Foiaş–Prodi theory) if and only if

$$(10)\qquad \int_K \phi(u)\,d\mu_t(u) - \int_K \phi(u)\,d\mu^0(u) = \int_0^\tau \int_K \int_\Omega u(x)\,\Delta \frac{\delta\phi(u)}{\delta u(x)}\,dx\,d\mu_t(u)\,dt + \int_0^\tau \int_K \int_\Omega f(t, x)\frac{\delta\phi(u)}{\delta u(x)}\,dx\,d\mu_t(u)\,dt\,.$$

holds for all continuous $\phi\colon H_0^2(\Omega) \to C$ having a Fréchet–Volterra derivative $\frac{\delta\phi(u)}{\delta u(\cdot)}$ of class C^2, and all $\tau \in [0, T)$.

THEOREM 3. *The moment functions of the statistical solution of* (1) *satisfy the following equations*

$$(11)\qquad \frac{\partial}{\partial t} M_n(t, x_1, \ldots, x_n) = \sum_{i=1}^n \Delta M_n(t, x_1, \ldots, x_n) + \sum_{i=1}^n f(t, x_i) M_{n-1}(t, x_1, \overset{i}{\underset{\cdots}{\vee}}, x_n)\;{}^{(2)}$$

$n = 1, 2, \ldots$ *and the initial conditions*

$$(12)\qquad M_n(0, x_1, \ldots, x_n) = M_n^0(x_1, \ldots, x_n)$$

$n = 1, 2, \ldots,$ *where* M_n^0 *are the moment functions of the initial measure* μ_0.

Proof. We apply (10) with functionals of particular form. For any $\phi_n(x_1, \ldots, x_n)$ of class $C_0^2(\Omega)$ with respect to each variable we put

$$\phi^n(u) = \int_\Omega \ldots \int_\Omega \phi_n(x_1, \ldots, x_n) u(x_1) \ldots u(x_n)\,dx_1 \ldots dx_n\,.$$

It is well known that

$$\frac{\delta\phi(u)}{\delta u(x)} = \sum_{i=1}^n \int_\Omega \ldots \int_\Omega \phi_n(x_1, \overset{i}{\underset{\cdots}{\vee}}, x_n) u(x_1) \overset{i}{\underset{\cdots}{\vee}} u(x_n)\,dx_1 \overset{i}{\underset{\cdots}{\vee}} dx_n$$

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(2) The sign $\overset{i}{\vee}$ means that x_i is omitted.

(see [8]). Thus we obtain

$$(13)\qquad \frac{\partial}{\partial t}\int_K\int_\Omega\cdots\int_\Omega \phi_n(x_1,\dots,x_n)u(x_1)\dots u(x_n)\,dx_1\dots dx_n\,d\mu_t(u)$$

$$=\int_K\int_\Omega u(x)\Delta\sum_i\int\cdots\int\phi_n(x_1,\overset{i}{\overset{\vee}{\ldots}},x_n)u(x_1)\overset{i}{\overset{\vee}{\ldots}}u(x_n)\,dx_1\overset{i}{\overset{\vee}{\ldots}}dx_n\,d\mu_t(u)+$$

$$+\int_K\int_\Omega f(t,x)\sum_i\int_\Omega\cdots\int_\Omega\phi_n(x_1,\overset{i}{\overset{\vee}{\ldots}},x_n)u(x_1)\overset{i}{\overset{\vee}{\ldots}}u(x_n)\,dx_1\overset{i}{\overset{\vee}{\ldots}}dx_n\,d\mu_t(u)\,.$$

Owing to

$$|\phi_n(x_1,\dots,x_n)u(x_1)\dots u(x_n)|\leqslant c_2 \quad \text{a.e. on } \Omega^n$$

we can apply Fubini's theorem in the left side of (13) obtaining

$$\frac{\partial}{\partial t}\int_\Omega\cdots\int_\Omega\phi_n(x_1,\dots,x_n)M_n(t,x_1,\dots,x_n)\,dx_1\dots dx_n.$$

Finally from 2° and 3° we find that the left side of (13) is equal to

$$(14)\qquad \int_\Omega\cdots\int_\Omega\phi_n(x_1,\dots,x_n)\frac{\partial}{\partial t}M_n(t,x_1,\dots,x_n)\,dx_1\dots dx_n\,.$$

Now we transform the right side of (13). As for any fixed t

$$|\Delta f(t,x_i)\phi_n(x_1,\dots,x_n)u(x_1)\dots u(x_n)|\leqslant c_3$$

a.e. on Ω^n so it is equal to

$$(15)\qquad \sum_{i=1}^n\int_K\int_\Omega\cdots\int_\Omega\Delta\phi_n(x_1,\dots,x_n)u(x_1)\dots u(x_n)\,dx_1\dots dx_n\,d\mu_t(u)+$$

$$+\sum_{i=1}^n\int_K\int_\Omega\cdots\int_\Omega f(t,x_i)\phi_n(x_1,\dots,x_n)u(x_1)\overset{i}{\overset{\vee}{\ldots}}u(x_n)\,dx_1\overset{i}{\overset{\vee}{\ldots}}dx_n\,d\mu_t(u)$$

$$=\sum_{i=1}^n\int_\Omega\cdots\int_\Omega\Delta\phi_n(x_1,\dots,x_n)M_n(t,x_1,\dots,x_n)\,dx_1\dots dx_n+$$

$$+\sum_{i=1}^n\int_\Omega\cdots\int_\Omega f(t,x_i)\phi_n(x_1,\dots,x_n)M_{n-1}(t,x_1,\overset{i}{\overset{\vee}{\ldots}}x_n)\,dx_1\overset{i}{\overset{\vee}{\ldots}}dx_n$$

$$=\int_\Omega\cdots\int_\Omega\phi_n(x_1,\dots,x_n)\Big[\sum_{i=1}^n\Delta M_n(t,x_1,\dots,x_n)+$$

$$+\sum_{i=1}^n f(t,x_i)M_{n-1}(t,x_1,\overset{i}{\overset{\vee}{\ldots}}x_n)\Big]dx_1\dots dx_n\,.$$

From the equality (14) = (15) we clearly get (11) for almost all $x_1, \ldots, x_n$, but from 2° this holds true on all Ω^n and for all $t \in (0, T)$. Taking (10) with $t = 0$ and $\phi = \phi^n$ we get (12). This completes the proof of Theorem 3.

It is clear that the problem (11)–(12) has a unique solution for each n (see for example [9]).

5. Equivalence of two definitions of statistical solution.

THEOREM 4. *Let us denote by* $\{\mu_t'\}$, $t \in (0, T)$ *the statistical solution of the heat equation in the sense of Definition* 3 *and by* $\{\mu_t''\}$, $t \in (0, T)$ *in the sense of Definition* 5 *with the same initial measure. Then* $\mu_t' = \mu_t''$ *for* $t \in (0, T)$.

Proof. We shall show that the moment functions M_n' of μ_t' (formula (9)) satisfy (11) and (12). Indeed, we have

$$\left(\frac{\partial}{\partial t} - \sum_{i=1}^{n} \Delta_{x_i}\right) M_n' = \int - \sum_{i=1}^{n} |\eta_i|^2 e^{i\langle x,\eta\rangle} e^{-t\sum_{i=1}^{n}|\eta_i|^2} \times$$

$$\times \sum_{\alpha^s-} \tilde{M}^0(\eta_{\alpha_1}, \ldots, \eta_{\alpha_s}) \prod_{i=1}^{n-s} \left(\int_0^t \hat{f}(s, \eta_{\beta_i}) e^{|\eta_{\beta_i}|^2 s} ds \right) d\eta_1 \ldots d\eta_n +$$

$$+ \int e^{i\langle x,\eta\rangle} e^{-t\sum_{i=1}^{n}|\eta_i|^2} \sum_{\alpha^s-} \tilde{M}^0(\eta_{\alpha_1}, \ldots, \eta_{\alpha_s}) \times$$

$$\times \frac{\partial}{\partial t} \left[\prod_{i=1}^{n-s} \int_0^t \hat{f}(s, \eta_{\beta_i}) e^{|\eta_{\beta_i}|^2 s} ds \right] d\eta_1 \ldots d\eta_n +$$

$$+ \int \sum_{i=1}^{n} |\eta_i|^2 e^{i\langle x.\eta\rangle} e^{-t\sum_{i=1}^{n}|\eta_i|^2} \sum_{\alpha^s-} \tilde{M}^0(\eta_{\alpha_1}, \ldots, \eta_{\alpha_s}) \times$$

$$\times \prod_{i=1}^{n-s} \left[\int_0^t \hat{f}(s, \eta_{\beta_i}) e^{|\eta_{\beta_i}|^2 s} ds \right] d\eta_1 \ldots d\eta_n +$$

$$+ \left(\frac{\partial}{\partial t} - \sum_{i=1}^{n} \Delta_{x_i}\right) \int e^{i\langle x,\eta\rangle} e^{-t\sum_{i=1}^{n}|\eta_i|^2} \tilde{M}^0(\eta_1, \ldots, \eta_n) d\eta_1 \ldots d\eta_n,$$

where the sum $\sum_{\alpha^s-}$ is taken over all proper subsets α^s of $\{1, \ldots, n\}$. Obviously the last component is equal to zero and the first one reduces with the second. We have further

$$\left(\frac{\partial}{\partial t}-\sum_{i=1}^{n}\Delta_{x_i}\right)M_n' = \int e^{i\langle x,\eta\rangle}e^{-t\sum_{i=1}^{n}|\eta_i|^2}\sum_{\alpha^s}\tilde{M}^0(\eta_{\alpha_1},\ldots,\eta_{\alpha_s})\times$$
$$\times\sum_{k=1}^{n=1}\left\{\hat{f}(t,\eta_{\beta_k})e^{t|\eta_{\beta_k}|^2}\prod_{\substack{i=1\\ i\neq k}}^{n-s}\left[\int_0^t\hat{f}(s,\eta_{\beta_i})e^{|\eta_{\beta_i}|^2 s}ds\right]\right\}d\eta_1\ldots d\eta_n$$

$$(*)\left\{\begin{aligned}&=\sum_{\alpha^s}\sum_{k=1}^{n-1}\int e^{ix_{\beta_k}\eta_{\beta_k}}\hat{f}(t,\eta_{\beta_k})d\eta_{\beta_k}\underbrace{\int\ldots\int}_{n-1}[e^{i(x_1\eta_1+\overset{\beta_k}{\overset{\vee}{\ldots}}+x_k\eta_k)}\times\\ &\qquad\times e^{-t(|\eta_i|^2+\overset{\beta_k}{\overset{\vee}{\ldots}}+|\eta_k|^2)}\tilde{M}^0(\eta_{\alpha_1},\ldots,\eta_{\alpha_s})\prod_{\substack{j=1\\ j\neq k}}^{n-s}\int_0^t\hat{f}(s,\eta_{\beta_j})e^{|\eta_{\beta_j}|^2 s}ds]d\eta_1\overset{\beta_k}{\overset{\vee}{\ldots}}d\eta_n\end{aligned}\right.$$

but

$$\int e^{ix_{\beta_k}\eta_{\beta_k}}\hat{f}(t,\eta_{\beta_k})d\eta_{\beta_k}=f(t,x)\,.$$

We end the proof of the fact that M_n' satisfies (11) and (12) by observing that the last sum $(*)$ is equal to the following sum

$$\sum_{n=1}^{n}f(t,x_m)M_{n-1}(x_1,\overset{m}{\overset{\vee}{\ldots}},x_n)$$

$$(**)\left\{\begin{aligned}&=\sum_{u=1}^{n}f(t,x_m)\sum_{\substack{\alpha^s-\\ m\notin\alpha^s}}\int e^{i(x_1\eta_1\overset{m}{\overset{\vee}{\ldots}}x_n\eta_n)}e^{-t(|\eta_1|^2+\overset{m}{\overset{\vee}{\ldots}}+|\eta_n|^2)}\times\\ &\qquad\times\tilde{M}^0(\eta_{\alpha_1},\ldots,\eta_{\alpha_s})\prod_{\substack{j=1\\ s_j\neq m}}^{n-s}\int_0^t\hat{f}(s,\eta_{\beta_j})e^{|\eta_{\beta_j}|^2 s}ds\,d\eta_1\overset{m}{\overset{\vee}{\ldots}}d\eta_n\,.\end{aligned}\right.$$

At first we take any component of the sum $(*)$, i.e. we fix any $\alpha^s\subsetneq\{1,\ldots,n\}$ and $\beta_k\in\{1,\ldots,n\}\setminus\alpha^s$. Taking $m=\beta_k$, α^s unchanged we make the component of $(**)$ equal to this. Conversely we take any component of $(**)$ by fixing $m\in\{1,\ldots,n\}$ and $\alpha^s\subseteq\{1,\ldots,n\}$, $m\notin\alpha^s$. We take α^s unchanged, we put $\beta_k=m$ and the component of $(*)$ equal to this has been found. On the other side the unique solutions of (11) and (12) are the moment functions M_n'' of μ_t''. Hence $M_n'=M_n''$ for each n. We denote by χ the set of all linear combinations of the functions $\varphi\colon H_0^1(\Omega)\to R$ of the following form

$$\varphi(u)=u(x_1)\ldots u(x_n),\ \ \varphi(u)=1\,,$$

where x_i $(i=1,\ldots,n)$ are arbitrarily chosen from R^3. Clearly

$$\int_K\varphi(u)d\mu_t'(u)=\int_K\varphi(u)d\mu_t''(u)$$

holds good for all $\varphi\in\chi$. Obviously $(\chi,+,\cdot)$ is an algebra with 1 separating points of K. From the Stone–Weierstrass theorem we get

$$\int_K f(u)d\mu_t'(u)=\int_K f(u)d\mu_t''(u)$$

for all continuous $f\colon K\to R$. Thus $\mu_t'(\omega)=\mu_t''(\omega)$ for any ω-Borel subset of $H_0^1(\omega)$. This finishes the proof of the Theorem.

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