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DXXX PRACE MATEMATYCZNE, ZESZYT 21 1979

A transversality property weaker than the Whitney *a*-regularity

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It is desired to decompose (stratify) a given real algebraic or analytic variety M in $\boldsymbol{R}^n$ into a finite disjoint union of C^1 manifolds $M = M_1 \cup \ldots \cup M_r$ in such a way that each stratum M_i consists of "equally good" points. For instance, let us consider the following vaiiesy in $\boldsymbol{R}^3$ (fig. 1). It consists of four cones having a common vertex and a common line. There is a stratification with five strata. Intuitively the vertex is "worse" than other points of M_5.

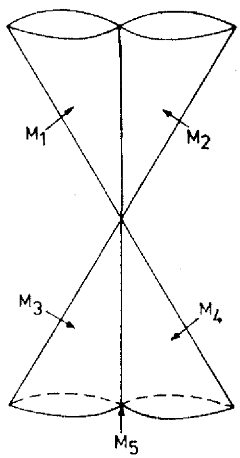

Fig. 1

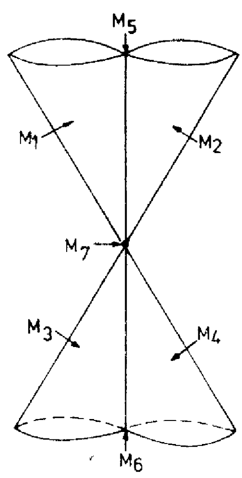

Fig. 2

The stratification in fig. 2 where the vertex alone forms the stratum M_7 looks regular. So we begin by introducing Whitney's regularity condition.

Definition. For two given C^1 submanifolds X, Y of $\boldsymbol{R}^n$ and a point $x \in X \cap \overline{Y}$ we say that *Y is (a) — regular over X at x* if the following holds:

(a) given any sequence $\{y_n\} \in Y$ such that $\lim\limits_{n\to\infty} y_n = x$ and $\lim\limits_{n\to\infty} T_{y_n} Y = \tau$ we have $T_x X \subset \tau$.

Here by $T_x Y$ we denote the tangent space to Y at x and convergence of the sequence $T_{y_n} Y$ means convergence in the standard topology in the Grassmanian manifold G_m of m-dimensional linear subspaces of $\mathbf{R}^n (m = \dim Y)$.

It is obvious that condition (a) implies the following condition

(t) $\forall C^1$-manifold M of $\mathbf{R}^n$ such that M is transversal at x to X $\exists$ a neighbourhood U of x such that M is transversal to Y in $Y \cap U$.

We recall that two submanifolds of $\mathbf{R}^n$ X, Y are transversal at x if $x \notin X \cap Y$ or $T_x X \oplus T_x Y = \mathbf{R}^n$.

D. Trotman has shown in [3] that the condition (t) does not imply (a).

In this paper we prove the following theorem

THEOREM. *If Y is a subanalytic manifold the condition* (t) *implies the condition* (a).

Definition $E \subset \mathbf{R}^n$ is *subanalytic* if $\forall x \in \mathbf{R}^n$ $\exists$ a neighbourhood U of x $\exists$ a semi-analytic set $A \subset \mathbf{R}^p$ for some p such that $E \cap U = \Pi(A)$, where we denote by Π the canonical projection from $\mathbf{R}^p$ on the first n-th coordinates.

We recall that $A \subset \mathbf{R}^p$ is *semi-analytic* if $\forall x \in \mathbf{R}^p$ $\exists$ a neighbourhood U of x such that $A \cap U = \bigcup_{i=1}^{s} (\bigcap_{j=1}^{r} \{g_{ij} > 0\} \cap \{f_i = 0\})$ where g_{ij}, f_i are analytic in U.

Proof of the Theorem. Suppose the condition (a) fails at $x \in X \cap \bar{Y}$. We shall give a construction of C^1 manifold M for which the condition (t) fails.

Choose a unit vector $v \in T_x X$ and a sequence $\{y_n\} \in Y$ such that $\lim_{n \to \infty} y_n = x$ and $\lim_{n \to \infty} T_{y_n} Y = \tau$ and $v \notin \tau$. Then $\exists \varepsilon > 0$ $\exists$ a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\forall_k d(v, T_{y_{n_k}} Y) > \varepsilon \tag{1}$$

(d denotes the standard distance in $\mathbf{R}^n$).

Let

$$V_1 = \mathbf{R}^n \times \{P \in G_m : d(v, P) > \varepsilon\} \subset \mathbf{R}^n \times G_m$$

and

$$V_2 = \bigcup_{y \in Y} (y, T_y Y) \subset \mathbf{R}^n \times G_m .$$

By its description V_1 is semi-algebraic and by our assumption on Y V_2 is sub-analytic. From (1) we get $(x, \tau) \in \overline{V_1 \cap V_2}$. We take a closed disc K centred at (x, τ) such that $K \cap \overline{V_1 \cap V_2} = \Pi(A)$ with Π and A as in Definition of a sub-analytic set. We may assume that A is a closed set because $\overline{\Pi(A)} \subset \overline{\Pi(\bar{A})} = \Pi(\bar{A}) \subset \overline{\Pi(A)} = \Pi(A)$ so $K \cap \overline{V_1 \cap V_2}$ is the projection of $\bar{A}$.

We take any point $z \in \Pi^{-1}(x, \tau) \cap A$. For z and A we use the curve selecting Lemma [1] and from Theorem 1 p. 127 in [1] we obtain an analytic arc in $\mathbf{R}^n \times G_m$ $\alpha\colon [0, \delta] \to \overline{V_1 \cap V_2}$ with $\alpha(0) = (x, \tau)$ $\alpha(t) \in V_1 \cap V_2$ if $t > 0$ such that $\alpha_{|(0, \delta+\delta')}$ is analytic embedding and there exists $\lim_{t \to 0} T_{\alpha(t)} \alpha([0, \delta])$. $\alpha(t) = (\gamma(t), T_{\lambda(t)} Y)$, where $\gamma(t)$ is the $\mathbf{R}^n$ component of $\alpha(t)$.

Let $N_t \in G_{n-1}$ for $t > 0$ be the normal space at $\gamma(t)$ to the C^1-manifold with boundary $\gamma([0, \delta])$ and $N_0 = \lim_{t \to 0} N_t$. Let us denote by v_t the orthogonal projection of v into N_t. Observe that $v_t \neq 0$ and hence it generates the one dimensional subspace $\langle v_t \rangle$.

Let $P_t = N_t \cap T_{\lambda(t)} Y$. Let us define $\sigma(t) = P_t \oplus (P_t \oplus \langle v_t \rangle)^{\perp}$ where $(\ )^{\perp}$ denotes the orthogonal complement in N_t. σ: $[0, \delta] \to G_{n-2}$ is a C^1-arc such that

$$\sigma(t) \oplus \langle v_t \rangle = N_t . \tag{2}$$

Then the union of the $\sigma(t)$ considered as embedded $n-2$ planes in $\boldsymbol{R}^n$ passing through the points $\gamma(t)$ defines a C^1-manifold with boundary M'. By reflection in N_0 we extend M' to a C^1-manifold M which is transversal to X at $x \in \operatorname{int} M$ by (2).

Given any neighbourhood U of x $\exists s \in (0, \delta)$ such that $U \cap \gamma(0, \delta] \supset \gamma(0, s]$ M' and hence M is not transversal to Y at any point of $\gamma(0, \delta]$, which completes the proof.

References

[1] S. Łojasiewicz, *Ensembles semi-analitiques*, I.H.E.S., Bures-sur-Yvette, 1965.

[2] J. Mather, *Notes on topological stability*, Oxford University Press, 1970.

[3] D. Trotman, *Whitney a-regularity*, Bull. London Math. Soc., 8, 1976, 225-228

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