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On the dimension of a sub-analytic set

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The aim of this paper is to give a uniform approach to the known definitions of the dimension of a sub-analytic set. Since authors of papers on sub-analytic sets give different definitions of the dimension, we decided to prove that these definitions are equivalent.

We use here the partition of a sub-analytic set into a countable union of submanifolds. Therefore we do not need to use any stratification of a sub-analytic set, proofs of the existence of such a stratification being fairly complicated (see [3], [4]).

We shall first give the definition of the sets we are concerned with:

Definition. A subset E of $\mathbf{R}^n$ is called *sub-analytic* if for every $x \in \bar{E}$ there is a neighbourhood U_x of x and a semi-analytic, relatively compact set A_x in $\mathbf{R}^{n+k_x}$ such that $E \cap U_x = \Pi(A_x)$, where $\Pi\colon \mathbf{R}^{n+k_x} \to \mathbf{R}^n$ is the projection.

It is easy to see that if the sub-analytic set $E \subset \mathbf{R}^n$ is relatively compact, there exists a semi-analytic set $A \subset \mathbf{R}^{n+k}$(with certain k) such that $E = \Pi(A)$, $\Pi\colon \mathbf{R}^{n+k} \to \mathbf{R}^n$ denotes the projection. We shall consider for the sake of simplicity only the case of a relatively compact sub-analytic set.

We begin with the following two simple lemmas:

LEMMA. *Let V, W, H be finite dimensional vector spaces, S and T linear transformations $S\colon V \to W$, $T\colon V \to H$ then $rkT_{|\ker S} = rk(S,T) - rkS$.*

Proof. The required formula follows immediately from the well-known equalities:

$$\ker(S,T) = \ker S \cap \ker T,$$
$$rk(S,T) = \dim V - \dim\ker(S,T),$$
$$rk S = \dim V - \dim\ker S,$$

hence

$$rkT_{|\ker S} = \dim\ker S - \dim\ker(T_{|\ker S}) = \dim\ker S - \dim(\ker S \cap \ker T)$$
$$= (\dim V - rk S) - (\dim V - rk(S,T)) = rk(S,T) - rk S.$$

LEMMA 2. *Let $\psi\colon G - \mathbf{R}^k$ be an analytic function defined on an open subset G of $\mathbf{R}^n$, such that $rk_a\psi = k$. Let us put $\Gamma = \{x \in G\colon \psi(x) = 0\}$ and let F be an analytic function defined on a neighbourhood of $\bar{\Gamma}$ with values in $\mathbf{R}^p$; then $rkF_{|\Gamma\cap U_a} = rk_a(F,\psi) - k$,* where U_a is a neighbourhood of a such that $rk_x\psi = k$ in U_a.

Proof. $\Gamma \cap U_a$ is an analytic submanifold with $\ker d_a\psi$ as the tangent space in a. Thus the required equality follows from Lemma 1.

Remark. From now on we shall use the notions of [1], especially those of normal neighbourhoods, partitions and distinguished polynoms. Let us recall that the system of distinguished polynoms must fulfil the two following implications:

(i) $H_k^{k-1} = H_l^k = 0 \Rightarrow H_l^{k-1} = 0$,

(ii) $H_l^k = \dfrac{\partial H_l^k}{\partial z_l} = 0 \Rightarrow H_k^{k-1} = 0$

in a certain neighbourhood of 0 in C^n.

PROPOSITION. *Let A be a relatively compact semi-analytic subset of $\boldsymbol{R}^n \times R^k$ and let $\Pi\colon \boldsymbol{R}^n \times \boldsymbol{R}^k \to \boldsymbol{R}^n$ be the projection. Then A admits a finite partition $A = U\Gamma_i$ such that Γ_i are members of a certain normal partition and $rk_x \Pi_{\Gamma_i}$* =const.

Proof. The proof goes by induction on $\dim A$. If $\dim A = 0$ the proposition is obvious. We shall prove that $A = A_1 \cup A_2$ where A_1, A_2 are semi-analytic, $\dim A_1 < \dim A$ and A_2 admits the required partition. (*)

For each $a \in \bar{A}$ we take a normal partition in a, compatible with A, of a normal neighbourhood Q_a such that any other normal partition in a compatible with A of a smaller neighbourhood of a is induced by that of Q_a (see [1]). Now we take a normal neighbourhood V_a such that $\bar{V}_a \subset Q_a$. Since $\bar{A}$ is compact, we have $A \subset V_{a_1} \cup \ldots \cup V_{a_k}$. Hence $A = UE_i$ (finite union), where E_i are members of a normal partition.

Let us consider a member E of this union such that $\dim E = \dim A = s$. It is easy to see that it suffices to prove (*) only for such a member.

E is a connected component of the manifold

$$V^s = \{x \in V\colon H_n^{n-1}(x) = \ldots = H_{s+1}^s(x),\ H_s^{s-1} \neq 0\} \quad \text{(see [1])},$$

V being a previously obtained V_{a_i}. Notice that for each $x \in V^s$ there exists a neighbourhood G_x of x such that

$$V^s \cap G_x \{y \in G_x\colon H_{s+1}^s(y) = \ldots = H_n^s(y) = 0\}$$

and

$$rk_y(H_{s+1}^s, \ldots, H_n^s) = n-s \quad \text{for } y \in G_x .$$

G_x may be defined, for instance, as an intersection $U_x \cap U'_x$ where U_x is a neighbourhood of x such that $U_x \cap V^s = U_x \cap C^s$ (V^s is open in $C^s = \{y \in V\colon H_{s+1}^s(y) = \ldots = H_n^s(y), H_s^{s-1}(y) \neq 0\}$) and U'_x is a neighbourhood of x such that $H_s^{s-1} \neq 0$ in U'_x. It follows from (i), (ii) that $rk\ (H_{s+1}^s, \ldots, H_n^s) = n-s$ in G_x.

Now let us consider the set $H = \{x \in V^s : rk_x \Pi_{V^s} = l\}$ where

$$l = \max\{rk_x \Pi_E\colon x \in E\} = \max\{rk_x \Pi_{V^s}\colon x \in E\} .$$

We claim that H is semi-analytic.

Observe first that $H = \{x \in V^s\colon rk_x(\Pi, H_{s+1}^s, \ldots, H_n^s) = l+n-s\}$. This follows from Lemma 2 and from the existence of an appropriate neighbourhood G_x as above. Since V^s is semi-analytic and the functions $\Pi, H_{s+1}^s, \ldots, H_n^s$ are analytic in a neighbourhood of $\bar{V}_s$, H and $H \cap E$ are also semi-analytic. Observe that

$$H \cap E = \{x \in E\colon rk_x \Pi_{V^s} = l\} = x \in E\colon rk_x \Pi_E = l\} \quad (E \text{ is open in } V^s) .$$

Hence $H \cap E$ is open in E and $H \cap E$ is an analytic submanifold of $\boldsymbol{R}^n$. From the description of $H \cap E$ and from the fact that E is a member of a normal partition (therefore E is a connected analytic submanifold) it follows that $H \cap E$ is dense in E. Thus $\dim(E \setminus H \cap E) < \dim E$.

Now for each $a \in \overline{H \cap E}$ we take a normal partition in a compatible with $H \cap E$. After choosing the finite number of these we get $H \cap E$ as a finite union of members of normal partitions. We define A_2 as union of those members which are open in $H \cap E$ and A_1 as a union of those which are not open in $H \cap E$ and of $E \setminus (H \cap E)$.

PROPOSITION 2. *Let E be a sub-analytic relatively compact set contained in $\boldsymbol{R}^n$.* Then E is a union of countable quantity *of analytic submanifolds of $\boldsymbol{R}^n$.*

Proof. This follows immediately from Proposition 1, from the rank theorem ([2]) and from the Lindelöf property of $\boldsymbol{R}^n$.

We shall need the following lemma:

LEMMA 3. *If E is a sub-analytic relatively compact subset of $\boldsymbol{R}^n$ such that* $\operatorname{int}\bar{E} \neq \emptyset$ *then* $\operatorname{int} E \neq \emptyset$ *also.*

Proof. We shall quote here the proof given by S. Łojasiewicz. Another proof may be found in ([5]).

Let A be a semi-analytic, relatively compact subset of $\boldsymbol{R}^{n+p}$ such that $E = \Pi(A)$, where Π denotes the projection $\Pi\colon \boldsymbol{R}^{n+p} \to \boldsymbol{R}^n$. By Proposition 1 we have a partition $\bar{A} = U\Gamma_i$ (finite union) such that Γ_i are members of a certain normal partition and $rk_x \Pi_{\Gamma_i} =$ const. Since $\operatorname{int}\bar{E} = \operatorname{int}\Pi(\bar{A}) \neq \emptyset$, there is Γ_{i_0} such that $rk\Pi_{\Gamma_{i_0}} = n$ (by the rank theorem, the Lindelöf property of $\boldsymbol{R}^{n+p}$ and the Baire theorem).

Let a be an arbitrary point of Γ_{i_0}. Let us take a normal partition in a compatible with Γ_{i_0}, A. Let Γ_0 be an open in Γ_{i_0} member of this partition. As $a \in \bar{A} \supset \Gamma_{i_0}$ there exists a member $\Gamma \subset A$ such that $\Gamma_0 \subset \bar{\Gamma}$. We shall show that $\Pi(\Gamma_0) \subset \Pi(\Gamma)$, and this will complete the proof, since $rk\Pi_{\Gamma_{i_0}} = rk\Pi_{\Gamma_0} = n$ implies $\operatorname{int}\Pi(\Gamma_0) \neq \emptyset$.

Let x belong to Γ_0. Notice that an affine space $x + (\{0\} \times \boldsymbol{R}^p)$ is transversal in x to Γ_0. By Proposition 4 from ([1]), p. 104, we get $(x + (\{0\} \times \boldsymbol{R}^p)) \cap \Gamma \neq \emptyset$. Hence $\Pi(x) \in \Pi(\Gamma)$, which completes the proof.

THEOREM. *Let Π be the projection $\Pi\colon \boldsymbol{R}^{n+p} \to \boldsymbol{R}^n$. Let E be a sub-analytic relatively compact subset of $\boldsymbol{R}^n$. The following definitions of the dimension of E are equivalent:*

Definition 1. $\dim E = \max\{\dim \sigma : \sigma$ an analytic submanifold, $\sigma \subset E\}$.

Definition 2. If $E = \bigcup_{i=1}^{\infty} \bigwedge_i$, where $\bigwedge_i$ are analytic submanifolds of $\boldsymbol{R}^n$ then $\dim E = \max \dim \bigwedge_i$.

Definition 3. If $E = \Pi(A)$, $A = A_1 \cup \ldots \cup A_r$ where A_i are semi-analytic connected analytic submanifolds of $\boldsymbol{R}^{n+p}$ such that $rk\Pi_{A_i} =$ const then $\dim E = \max_{i=1,\ldots,r} rk\Pi_{A_i}$.

Definition 4. If $E = \Pi(A)$, where A is a semi-analytic subset of $\boldsymbol{R}^{n+p}$,

$$A^0 = \{x \in A\colon x \text{ is a regular point of } A\}$$

then $\dim E = \max\{rk_x \Pi_{A \cap U_x}\colon x \in A^0\}$ (U_x is a neighbourhood of x such that $A \cap U_x$ is an analytic submanifold).

Definition 5. $\dim E = \max\{k$: there exists L — a linear k-dimensional subspace of $\boldsymbol{R}^n$ such that $\operatorname{int}\Pi_k(E) \neq \emptyset\}$ (Π_k denotes the orthogonal projection $\Pi_k : \boldsymbol{R}^n \to L$).

Definition 5 is given by Gabrielow, Definition 1 is given by Łojasiewicz and it is easy to see that Definition 2 is equivalent to that of Hardt ([3]).

We shall prove the sequence of inequalities $1 \geqslant 2 \geqslant 3 \geqslant 4 \geqslant 5 \geqslant 1$, where, for instance, $1 \geqslant 2$ means that $\dim E$ obtained by Definition 2 is not greater than the obtained by Definition 1.

Proof. Inequality $1 \geqslant 2$ is obvious.

$2 \geqslant 3$. Let us assume that $E = \Pi(A)$ and $A = A_1 \cup \ldots \cup A_r$ is the partition as in Definition 3. It follows from the rank theorem that for each $x \in A_i$ there is a neighbourhood U_x of x such that $\Pi(U_x \cap A_i)$ is a submanifold of $\boldsymbol{R}^n$. Choosing a countable quantity of U_x covering A we get $E = U\Pi(U_x^j \cap A_i)$. Since $\dim \Pi(U_x \cap A_i) = rk_x \Pi_{A_i}$, we obtain the required inequality.

$3 \geqslant 4$. Let a be a point of A^0 such that $rk_a \Pi_{A \cap U_a} = \max\{rk_x \Pi_{U_x \cap A}\, x \in A^0\}$. Since A^0 is open in A, there is a neighbourhood U_a of a such that $U_a \cap A^0 = U_a \cap A$. Taking U_a smaller, if needed, we may assume that $rk\Pi_{U_a \cap A^0} = rk\Pi_{U_a \cap A} =$ const and $U_a \cap A$ is connected. Thus there exists A_i from the partition of A such that $U_a \cap A \subset A_i$. Hence we obtain $rk\Pi_{A \cap U_a} \leqslant rk\Pi_{A_i} \leqslant \max\limits_{i=1\ldots r} \Pi_{A_i}$.

$4 \geqslant 5$. Let L_k be a linear k-dimensional subspace of $\boldsymbol{R}^n$ such that $\operatorname{int} \Pi_k(E) \neq \emptyset$. Let A be a semi-analytic subset of $\boldsymbol{R}^{n+p}$ such that $E = \Pi(A)$. Observe that

$$\emptyset = \operatorname{int} \Pi_k(E) = \operatorname{int} \Pi_k(\Pi(A)).$$

From lemma 3 and from the equality $\overline{A^0} = A$ ([1]) it follows that $\operatorname{int} \Pi_k(\Pi(A^0)) \neq \emptyset$, too. From Proposition 1 we get the partition $A^0 = U\Gamma_i$ (finite union), where Γ_i are members of a normal partition, $rk_x(\Pi_k \circ \Pi)_{\Gamma_i} =$ const. If $rk_x(\Pi_k \circ \Pi)_{\Gamma_i}$ were strongly smaller than k for every i, $\operatorname{int} \Pi_k \circ \Pi(A^0)$ would be empty (by the rank theorem, the Lindelöf property of $\boldsymbol{R}^{n+p}$ and the Baire theorem). So there is Γ_{i_0} such that $rk_x(\Pi_k \circ \Pi)_{\Gamma_{i_0}} = k$ for $x \in \Gamma_{i_0}$. Let a belong to Γ_{i_0}. We take U_a a neighbourhood of a such that $A^0 \cap U_a$ is a submanifold of $\boldsymbol{R}^{n+p}$. We have the obvious inequalities

$$rk_a\, \Pi_{A^0 \cap U_a} \geqslant rk_a(\Pi_k \circ \Pi)_{A^0 \cap U_a} \geqslant rk_a(\Pi_k \circ \Pi)_{i_0 \cap U_a} = k\,.$$

$5 \geqslant 1$. If σ_0 is an analytic submanifold of $\boldsymbol{R}^n$ contained in E such that $\dim \sigma_0 = \max\{\dim \sigma : \sigma \subset E,\ \sigma$ an analytic submanifold of $\boldsymbol{R}^n\}$, we may take its tangent space on an arbitrary point and the projection of σ_0 on it (let alone the projection of E) has a non-empty interior.

Remark. From the fact that the definitions we are concerned with are equivalent to each other it follows that Definition 2 does not depend on the partition of E into submanifolds, Definition 3 does not depend on the choice of A, A_i and Definition 4 does not depend on the choice of A.

Remark. It is easy to obtain the following useful equality from Definition 5 and Lemma 3: $\dim E = \dim \bar{E}$ for E a sub-analytic subset of $\boldsymbol{R}^n$.

References

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