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Construction of an orthonormal basis in the space of functions analytic in a disc and of class C^n in its closure

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1. Introduction. Let $\Delta = \{-\pi = t_0 < t_1 < \ldots < t_N = \pi\}$ be a partition of the interval $T = [-\pi, \pi]$. The function $s_\Delta \in C^{m-1}(T)$ is called a *spline of degree m with respect to the partition* Δ, where $m \geq 1$, if it is in each interval $[t_{i-1}, t_i]$ a polynomial of degree at most m. The spline s_Δ is said to be *periodic* of period 2π if $s_\Delta^{(j)}(-\pi) = s_\Delta^{(j)}(\pi)$ for $j = 0, 1, \ldots, m-1$.

Let $A_n = A_n(D)$ be the Banach space of analytic functions in the unit disc $D = \{z: |z| < 1\}$ which have n continuous derivatives in $\bar{D}$ with the norm

$$\|f\|^{(n)} = \sum_{j=0}^{n} \|f^{(j)}\|, \quad \text{where } \|f^{(j)}\| = \max_{|z|=1} |f^{(j)}(z)|.$$

In this space the following scalar product is given

$$(f, g) = \int_{-\pi}^{\pi} f(e^{it})\overline{g(e^{it})}\,dt.$$

The function S_Δ defined by means of the Schwarz formula

$$(1) \qquad S_\Delta(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_\Delta(t) \frac{e^{it}+z}{e^{it}-z}\,dt + iA, \ |z|<1, \quad \text{where } A = \text{const}$$

is said to be an *analytic spline of degree m* associated with the function s_Δ. On the circle $\Gamma = \{z: |z| = 1\}$ we set

$$(2) \qquad S_\Delta(e^{i\varphi}) = s_\Delta(\varphi) + \frac{i}{2\pi} \int_{-\pi}^{\pi} s_\Delta(\varphi - t)\operatorname{ctg}\frac{t}{2}\,dt + iA,$$

where the integral is interpreted as Cauchy Principal Values [7], [10]. Because the function s_Δ is of class $C^{m-1}(T)$ the analytic spline S_Δ belongs to the space A_{m-1}.

The sequence $\{f_k\}_{k=0}^{\infty}$ of elements of a given Banach space X is called a *basis* whenever each $f \in X$ has a unique expansion

$$f = \sum_{k=0}^{\infty} a_k f_k$$

convergent in the norm.

A basis $\{f_k\}_{k=0}^{\infty}$ in the space A_n is called *simultaneous* if the system $\{f_k\}_{k=0}^{\infty}$ is a basis in the space A_j for $j = 0, 1, \ldots, n$.

S. V. Bočkarev [3], [4] constructed an orthonormal basis $\{G_k\}_{k=0}^{\infty}$ in the space A_0. He solved the Banach problem of existence of a basis in this space over fourty years ago. Afterwards Z. Ciesielski generalized Bočkarev's result, constructing a simultaneous basis in the space A_n $(n>0)$ [5], [6], but his basis is not orthogonal.

Applying (1), the fundamental identity for splines of odd degree and the results of J. H. Ahlberg, E. N. Nilson, J. L. Walsh [1], Z. Ciesielski [5], [6] and Z. Wronicz [9] we shall construct orthonormal bases in the space A_n.

2. Orthonormal systems in the space A_n. Let $f \in A_n$, $f(e^{it}) = u(t)+iv(t)$, $-\pi \leq t \leq \pi$. Dispose the functions u and v on the even and odd parts, $u = u_1+u_2$, $v = v_1+v_2$, where $u_1(t) = \frac{1}{2}[u(t)+u(-t)]$, $u_2(t) = \frac{1}{2}[u(t)-u(-t)]$ and dispose analogously the functions v_1 and v_2. The periodic functions u_1 and v_1 satisfy the following conditions:

$$u_1^{(2j-1)}(0) = u_1^{(2j-1)}(\pi) = v_1^{(2j-1)}(0) = v_1^{(2j-1)}(\pi) = 0, \quad 1 \leq 2j-1 \leq n\,. \tag{3}$$

The periodic functions u_2 and v_2 satisfy the following conditions:

$$u_2^{(2j)}(0) = u_2^{(2j)}(\pi) = v_2^{(2j)}(0) = v_2^{(2j)}(\pi) = 0, \quad 0 \leq 2j \leq n\,. \tag{4}$$

Let P_n and N_n be the spaces of functions of class C^n in the interval $[0, \pi]$ satisfying the conditions (3) or (4) respectively with the following scalar product: $(f, g) = \int_0^{\pi} f(t)g(t)\,dt$. Now we shall construct orthonormal systems in these spaces. For the purpose we need the following

THEOREM 1. *Let* $\Delta = \{0 = t_0 < t_1 < \ldots < t_N = \pi\}$, $(N \geq 2)$ *be a given partition of the interval* $[0, \pi]$. *If*

1^0 $f \in P_n$ *and* s_Δ *is a spline of degree* $2n+3$ *of interpolation to* f *on* Δ *(i.e.* $s_\Delta(t_j) = f(t_j)$, $j = 0, 1, \ldots, N$*) satisfying the following conditions*:

$$s_\Delta^{(n\pm i)}(0) = s_\Delta^{(n\pm i)}(\pi) = 0, \quad 1 \leq i = 2j-1 \leq n\,, \tag{5}$$

or

2^0 $f \in N_n$ *and* s_Δ *is a spline of degree* $2n+3$ *of interpolation to* f *on* Δ *satisfying the following conditions*:

$$s_\Delta^{(n\pm i)}(0) = s_\Delta^{(n\pm i)}(\pi) = 0, \quad 0 \leq i = 2j \leq n\,, \tag{6}$$

then the spline s_Δ *exists and is unique for arbitrary function* f.

Proof. We may write the function s_Δ as follows:

$$s_\Delta(t) = a_0 + a_1 t + \ldots + a_{2n+3} t^{2n+3} + \sum_{j=1}^{N-1} \lambda_j (t-t_j)_+^{2n+3}\,, \tag{7}$$

where $(t-t_j)_+^{2n+3} = \max\big(0, (t-t_j)\big)^{2n+3}$.

By assumption we obtain the system of $2n+N+3$ equations and $2n+N+3$ unknown quantities. We shall prove that the matrix of this system is nonsingular. To do this it suffices to prove that $s_\Delta = 0$ for $f = 0$.

Analogously as in [1, pp. 154–160] we deduce that the first integral relation holds true for the functions f and s_Δ, i.e.

(8) $$\int_0^\pi [f^{(n+2)}(t)]^2\,dt = \int_0^\pi [s_\Delta^{(n+2)}(t)]^2\,dt + \int_0^\pi [f^{(n+2)}(t) - s_\Delta^{(n+2)}(t)]^2\,dt\,.$$

It follows from this relation that $s_\Delta = 0$ for $f = 0$.

LEMMA. *Let $\Delta_1 \subset \Delta_2$ be two partitions of the interval $[0, \pi]$. If s_1 and s_2 are splines of degree $2n+3$ with respect to the partitions Δ_1 and Δ_2 respectively satisfying* (5) *or* (6) *respectively such that $s_2(t) = 0$ for $t \in \Delta_1$, then*

$$\int_0^\pi s_1^{(n+2)}(t)\, s_2^{(n+2)}(t)\,dt = 0\,.$$

Proof. Let $\Delta_1 = \{0 = t_0 < \ldots < t_N = \pi\}$. We integrate $n+2$ times by parts and we obtain

$$\int_0^\pi s_1^{(n+2)}(t)s_2^{(n+2)}(t)\,dt = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} s_1^{(n+2)}(t)s_2^{(n+2)}(t)\,dt$$
$$= \sum_{i=1}^{N}\Big(\sum_{k=1}^{n+2} (-1)^{k+1} s_2^{(n-k+2)}(t)s_1^{(n+k+1)}(t)\Big|_{t_{i-1}}^{t_i}\Big) = 0\,.$$

Let $\{\Delta_N\}_{N=1}^\infty$ be a given sequence of partitions of the interval

$$J = [0, \pi],\ \Delta_N = \{0 = t_{N,0} < t_{N,1} < \ldots < t_{N,N} = \pi\}$$

with $\Delta_N \subset \Delta_{N+1}$. Define the following system of functions $\{\varphi_{N,n}\}_{N=2}^\infty$: $\varphi_{N,n}$ is a spline of degree $2n+3$ with respect to the partition Δ_N satisfying (5) equal to one on $\Delta_N \setminus \Delta_{N-1}$ and equal to zero on Δ_{N-1}. Then it follows from the lemma that the system $\{f_{N,n}\}_{N=1}^\infty$, where

$$f_{1,n} = \frac{1}{\sqrt{\pi}}, \quad f_{N,n} = \frac{\varphi_{N,n}^{(n+2)}}{\|\varphi_{N,n}^{(n+2)}\|_2}, \quad N > 1, \quad \|f\|_2 = \left(\int_0^\pi [f(t)]^2\,dt\right)^{1/2},$$

is orthonormal in the space P_n.

Define the following system of functions $\{F_{N,n}\}_{N=1}^\infty$:

$$F_{N,n}(t) = \begin{cases} f_{N,n}(t) & \text{for } t \in [0, \pi]\,, \\ f_{N,n}(-t) & \text{for } t \in [-\pi, 0]\,. \end{cases}$$

Let

(9) $$G_{N,n}(z) = \frac{1}{2\pi\sqrt{2}} \int_{-\pi}^{\pi} F_{N,n}(t)\,\frac{e^{it}+z}{e^{it}-z}\,dt, \qquad N = 1, 2, \ldots$$

The functions $G_{N,n}$ belong to the space A_n. Analogously as in [10] we can prove that the system $\{G_{N,n}\}_{N=1}^\infty$ is orthonormal in the space A_n.

As above, define the following system of functions $\{\psi_N\}_{N=1}^\infty$: $\psi_{N,n}$ is a spline of degree $2n+3$ with respect to the partition Δ_N satisfying (6) equal to one on $\Delta_N \setminus \Delta_{N-1}$ and equal to zero on Δ_{N-1}. Then the system $\{\hat{f}_{N,n}\}_{N=2}^{\infty}$, where $\hat{f}_{N,n} = \dfrac{\psi_{N,n}^{(n+2)}}{\|\psi_{N,n}^{(n+2)}\|_2}$ is orthonormal in the space N_n. Define the following system of functions:

$$\hat{F}_{N,n}(t) = \begin{cases} \hat{f}_{N,n}(t) & \text{for } t \in [0, \pi], \\ -\hat{f}_{N,n}(-t) & \text{for } t \in [-\pi, 0]. \end{cases}$$

As above, we prove that the system of functions

(10)
$$\hat{G}_{1,n}(z) = \frac{1}{\sqrt{2\pi}},$$
$$\hat{G}_{N,n}(z) = \frac{1}{2\pi\sqrt{2}} \int_{-\pi}^{\pi} \hat{F}_{N,n}(t) \frac{e^{it}+z}{e^{it}-z}\, dt, \quad N = 2, 3, \ldots$$

is orthonormal in the space A_n.

3. Orthonormal bases in the space A_n. Let $\{\Delta_N\}_{N=2}^{\infty}$ be a given sequence of partitions of the interval $J = [0, \pi]$, $\Delta_N = \{0 = t_{N,0} < \ldots < t_{N,N} = \pi\}$ with $\Delta_N \subset \Delta_{N+1}$ such that Δ_{2^k} is an equidistant partition of the interval J i.e. $t_{2^k,i} = \pi i/2^k$, $i = 0, 1, \ldots, 2^k$ and $t_{N,k+1} - t_{N,k} \geqslant \geqslant t_{N,k} - t_{N,k-1}$, $k = 1, 2, \ldots, N-1$.

Further we need the following

THEOREM 2. *Let Δ'_N be a partition of the interval $[-\pi, \pi]$ such that*

$$\Delta'_N = \{-\pi = -t_{N,N} < \ldots < -t_{N,1} < t_{N,0} = 0 < t_{N,1} < \ldots < t_{N,N} = \pi\}$$

($\Delta'_N \cap J = \Delta_N$) and let f be a periodic function of class C^k ($k \leqslant 2n+2$) in the interval $[-\pi, \pi]$ and s_N a periodic spline of degree $2n+3$ with respect to the partition Δ'_N of interpolation to the function f on $\Delta'_N(s_N(\pm t_j) = f(\pm t_j))$. Then there exists a constant α_n such that

(11)
$$\|f^{(i)} - s_N^{(i)}\| \leqslant \alpha_n \|\Delta'_N\|^{k-i} \omega(f^{(k)}, \|\Delta_N\|), \quad 0 \leqslant i \leqslant k \leqslant 2n+2.$$

Proof. The inequality (11) was proved in [5], [6] and [9, Theorem 4] for the partition Δ of the interval $[-\pi, \pi]$, $\Delta = \{-\pi = t_0 < t_1 < \ldots < t_N = \pi\}$ such that $t_{j+1} - t_j \geqslant t_j - t_{j-1}$, $j = 1, \ldots, N-1$. Because of the periodicity of the functions f and s_N we may consider the difference $f(t) - s_N(t)$ on an arbitrary interval of the length 2π. For $N = 2^m$ the partition Δ'_N is equidistant and the above condition is satisfied. Let $N = 2^m + r$, $0 < r < 2^m$ and let j be such that $t_{N,j-1} - t_{N,j-2} = t_{N,j} - t_{N,j-1} = \frac{1}{2}(t_{N,j+1} - t_{N,j})$. Now we consider this difference on the interval $[t_{N,j}, t_{N,j} - 2\pi]$. For this interval we obtain a partition which satisfies the above condition, and this establishes the theorem.

COROLLARY 1. *The system $\{f_{N,n}\}_{N=1}^{\infty}$ is an orthonormal basis in the space P_n and there exists a constant α_n such that for the function $f \in P_n \cap C^k([0, \pi])$, $0 \leqslant k \leqslant n$*

(12)
$$\|f^{(i)} - s_{N,f}^{(i)}\| \alpha_n \|\Delta_N\|^{k-i} \omega(f^{(k)}, \|\Delta_N\|), \quad 0 \leqslant i \leqslant k,$$

where $s_{N'f}$ is the N^{th} Fourier sum of the function f with respect to the system $\{f_{N,n}\}_{N=1}^{\infty}$.

Proof. Extend the functions f and $s_{N,f}$ to the functions $\tilde{f}$ and $\tilde{s}_{N,f}$ even and periodic in the interval $[-\pi, \pi]$. Let $\tilde{F}$ and $\tilde{S}$ be periodic functions in $[-\pi, \pi]$ such that $\tilde{F}^{(n+2)} = \tilde{f} - \tilde{s}_{1,f}$, $\tilde{S}^{(n+2)} = \tilde{s}_{N,f} - \tilde{s}_{1,f}$ and $\tilde{F}(-\pi) = S(-\pi)$. It follows from the first integral relation for periodic splines that the function $\tilde{S}$ is a periodic spline of degree $2n+3$ with respect to the partition Δ'_N of interpolation to the function $\tilde{F}$ on Δ'_N. The inequality (11) holds true for these functions and the inequality (12) is its particular case.

As above we obtain

COROLLARY 2. *The system* $\{\hat{f}_{N,n}\}_{N=2}^{\infty}$ *is an orthonormal basis in the space* N_n *and there exists a constant* α_n *such that for the function* $f \in N_n \cap C^k([0, \pi])$, $0 \leqslant k \leqslant n$

$$\| f^{(i)} - \hat{s}_{N,f}^{(i)} \| \leqslant \alpha_n \|\Delta_N\|^{k-i} \omega(f^{(k)}, \|\Delta_N\|), \quad 0 \leqslant i \leqslant k \,, \tag{13}$$

where $\hat{s}_{N,f}$ *is the* N^{th} *Fourier sum of the functions* f *with respect to the system* $\{\hat{f}_{N,n}\}_{N=2}^{\infty}$.

We obtain the next theorem on the basis of these corollaries.

THEOREM 3. *Let* $\{\Delta_N\}_{N=2}^{\infty}$ *be a given sequence of partitions of the interval* $J = [0, \pi]$ *with* $\Delta_N \subset \Delta_{N+1}$, *defined at the beginning of this point. Then the systems* (9) *and* (10) *defined for this sequence of partitions are orthonormal and simultaneous bases in the space* A_n *and there exists a constant* β_n *such that for* $f \in A_n$, $f(e^{it}) = u(t) + iv(t)$

$$\begin{aligned} &\| f^{(i)} - S_{N,f}^{(i)} \| \leqslant \beta_n \|\Delta_N\|^{n-i} [\omega(u^{(n)}, \|\Delta_N\|) + \omega(v^{(n)}, \|\Delta_N\|)] \,, \\ &\| f^{(i)} - \hat{S}_{N,f}^{(i)} \| \leqslant \beta_n \|\Delta_N\|^{n-i} [\omega(u^{(n)}, \|\Delta_N\|) + \omega(v^{(n)}, \|\Delta_N\|)] \,, \end{aligned} \tag{14}$$

where $S_{N,f}$ *and* $\hat{S}_{N,f}$ *are the* N^{th} *Fourier sums of the function* f *with respect to the systems* (9) *and* (10) *respectively.*

The proof is analogous to the proof of the fact that Bočkarev's system is a basis in the space A_0 (see [9]).

Remark. J. N. Subbotin constructed the system (9) in another way in [8], but he only proved that this system is a basis in the space A_0.

Let $A_n(D^k)$ be the Banach space of analytic functions f in the polydisc D^k which have continuous derivatives $\frac{\partial^{i_1+\ldots+i_k}}{\partial z_1^{i_1} \ldots \partial z_k^{i_k}} f$ for $i_1 + \ldots + i_k \leqslant n$ in $\overline{D^k}$ with the norm

$$\| f \|^{(n)} = \sum_{i \leqslant n} \left\| \frac{\partial^{i_1+\ldots+i_k}}{\partial z_1^{i_1} \ldots \partial z_k^{i_k}} f \right\|, \quad \text{where } i = i_1 + \ldots + i_k$$

and

$$\| f \| = \max\{|f(z_1, \ldots, z_k)|, (z_1, \ldots, z_k) \in \overline{D^k}\} \,.$$

As a conclusion from Theorem 3 we obtain

THEOREM 4. *The systems* $\{G_{N_1,n}(z_1) \cdot \ldots \cdot G_{N_k,n}(z_k)\}_{N_j=1}^{\infty}$ and $\{\hat{G}_{N_1,n}(z) \cdot \ldots \cdot \hat{G}_{N_k,n}(z_k)\}_{N_j=1}^{\infty}$, $j = 1, 2, \ldots, k$ *are orthonormal and simultaneous bases in the space* $A_n(D^k)$ *with respect to the scalar product* $(f, g) = \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} f(e^{it_1}, \ldots, e^{it_k}) \overline{g(e^{it_1}, \ldots, e^{it_k})} dt_1 \ldots dt_k$.

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