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# Statistical Solutions of the Heat Equation

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**1. Introduction.** In the theory of non-stationary initial problems for partial differential equations we examine the time development of the given initial value. However many problems in mechanics (particularly in fluid mechanics) show that the following approach is more useful. We are given the probability measure on the space of all initial values (so called initial measure). We consider the time evolution of this measure according to the given differential equation. In this way we obtain the family of probabilities (so called statistical solution). In the case of the Navier–Stokes equations this approach has been developed by C. Foiaş in [1].

In this paper we consider the heat transfer equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x)-\Delta_x u(t,x)=f(t,x) & \text{for } t\in(0,T),\ x\in\Omega\subset R^n,\\ u(0,x)=u_0(x) & \text{for } x\in\Omega,\\ u(t,x)=0 & \text{for } t\in[0,T),\ x\in\delta\Omega \end{cases}$$

assuming that initial value and right side term are both random (i.e., we have two probability measures $\mu_0$ and $\zeta$ on the space of all initial values and right side terms respectively). We give the appriopriate definition of the statistical solution and compare it with the statistical solution of the heat equation defined (in [2]) analogically like in [1]. After few auxiliary theorems we give the existence and uniqueness theorems.

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**2. Basic definitions and notations.** Let $L^2$ denote the usual Hilbert space of all square integrable real valued functions defined on the bounded domain $\Omega\subset R^n$ with boundary of class $C^2$. The scalar product of $u, v\in L^2$ will be denoted by $(u, v)$ and the norm of $u\in L^2$ by $||u||$. $H_0^1$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $||u||_1 := (u, u)_1^{1/2}$, where

$$(u, v)_1 := \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right).$$

$H_0^1$ with above scalar product is the Hilbert space. The Friedrichs selfadjoint extension of the operator $-\Delta$ (considered on the space $C_0^\infty(\Omega)$) will be denoted by $A$. We have

$$D(A)\subset H_0^1, \quad D(A^{1/2}) = H_0^1,$$
$$(u, v)_1 = (A^{1/2}u, A^{1/2}v),$$
$$\|u\| \leqslant c_1 \|u\|_1$$

for $u, v \in H_0^1$ (see [3] p. 191, [4] p. 110, [5] p. 389). $H_0^{-1}$ denotes the conjugate space to the space $H_0^1$. It can be identified with the complement of the unitary space $(L^2, (\cdot,\cdot)_{-1})$, where $(u, v)_{-1} := (Iu, v)$ and the operator $I$: $L^2 \to H_0^1$ is determined by the following equality

$$(u, w) = (Iu, w)_1 \quad \text{for } w \in H_0^1.$$

The value of $u \in H_0^{-1}$ in the point $v \in H_0^1$ will be denoted by $u[v]$. We have

$$|u[v]| \leqslant \|u\|_{-1} \|u\|_1, \quad \|u\|_{-1} \leqslant c_1 \|u\|,$$

where $\|u\|_{-1} := (u, u)_{-1}^{1/2}$.
We can extend the operator $A$ to the operator $A_e$: $H_0^1 \to H_0^{-1}$. For $u \in H_0^1$ we put $A_e u[v] := (u, v)_1$ $(v \in H_0^1)$.

The following theorem has been proved in [2] (it is a modification of Theorem 2.2, 2.3 from [6]):

THEOREM 1. *For every* $u_0 \in L^2, f \in L^2(0, T; L^2)$ *there exists the unique* $u$: $(0, T) \to H_0^1$ *such that*

1) $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1)$,
2) $u$ *has a weak derivative* $u'$ *as a function from* $(0, T)$ *to* $H_0^{-1}$,
3) *for almost all* $t \in (0, T)$ *we have*

(1)
$$u'(t) + A_e u(t) = f(t),$$
$$u(t) \to u_0 \text{ weakly in } L^2 \text{ as } t \to 0,$$

4) *there exist constants* $c_2$, $c_3$, $c_4$, $c_5$ *such that*

$$\|u(t)\| \leqslant c_2 \|u\| + c_3 \|f\|_{L^2(0,T;L^2)},$$

(2)
$$\int_0^T \|u(t)\|_1^2 dt \leqslant c_4 \|u\|^2 + c_5 \|f\|^2_{L^2(0,T;L^2)}.$$

The function $u$ satisfying the above conditions will be called *weak solution of the heat equation.* To underline the dependence of $u$ on $u_0$ and $f$, this solution will be denoted by $S(t, u_0, f)$. The mappings

$$S: (0, T) \times L^2 \times L^2(0, T; L^2) \to L^2,$$
$$S: (0, T) \times H_0^1 \times L^2(0, T; L^2) \to H_0^1,$$

are both continuous with respect to each variable and with respect to the couple $(u_0, f)$ (see [6] p. 158).

**3. Definition of the statistical solution of the heat equation.** We shall use the following notations: $M := L^2 \times L^2(0, T; L^2)$, $N := H_0^1 \times L^2(0, T; L^2)$.

Definition 1. We say that $F: M \to R$ is the *test functional* if

1) it is a continuous function,

2) there exist constants $c_6$, $c_7$, $c_8$ such that

$$|F(u,f)| \leqslant c_6 + c_7 ||u|| + c_8 ||f||_{L^2(0,T;L^2)}$$

for all $(u,f) \in M$,

3) for arbitrary fixed $f$, for each $u$ there exists $F_1'(u,f) \in H_0^1$ such that

$$\frac{|F(u+v,f) - F(u,f) - (F_1'(u,f), v)|}{||v||} \to 0 \quad \text{if } ||v|| \to 0,\ v \in H_0^1,$$

4) the mapping $F_1': M \to H_0^1$ is continuous and for some constant $c_9$ we have

$$||F_1'(u,f)||_1 \leqslant c_9$$

for all $(u,f) \in M$.

Definition 2. The family $\{\nu_t\}_{t \in (0,T)}$ of Borel, probability measures on $M$ will be called *basic* if

1) $\nu_t(N) = 1$ for each $t \in (0, T)$,

2) the function

$$t \to \int_N \{||u||_1^2 + ||f(t)||^2\} d\nu_t(u,f)$$

belongs to $L^1(0, T; R)$,

3) for some constant $c_{10}$ we have

$$\int_N \{||u|| + ||f||_{L^2(0,T;L^2)}\} d\nu_t(u,f) \leqslant c_{10}$$

for $t \in (0, T)$.

We fix two Borel probabilities $\mu_0$ and $\zeta$ on the space $L^2$ and $L^2(0, T; L^2)$ respectively. We assume that

$$\int_{L^2} ||u||^2 d\mu_0(u) < +\infty$$

$$\int_{L^2(0,T;L^2)} ||f||_{L^2(0,T;L^2)} d\zeta(f) < +\infty .$$

Definition 3. The basic family of measures $\{\nu_t\}_{t \in (0,T)}$ will be called a *statistical solution of the heat equation* corresponding to the given measures $\mu_0$ and $\zeta$ if the following equality holds for each test functional $F$ and for each $s \in (0, T)$:

$$(3) \quad \int_N F(u,f) d\nu_s(u,f) - \int_M F(u,f) d\mu_0 \times \zeta(u,f) + \\ + \int_0^s \int_N (u, F_1'(u,f))_1 d\nu_t(u,f) dt = \int_0^s \int_N (f(t), F_1'(u,f)) d\nu_t(u,f) dt .$$

The notion of the statistical solution in the sense of the above definition generalizes the notion of the statistical solution presented in [2]. The following theorem describes this connection precisely.

THEOREM 2. *The conditions* (i) *and* (ii) *are equivalent*:

(i) *the family of measures* $\{\mu_t\}_{t\in(0,T)}$ *is the statistical solution of the heat equation corresponding to the initial measure* $\mu_0$ *and fixed right side term* $f \in L^2(0, T; L^2)$ *in the sense of definition* 4 *from* [2],

(ii) *the family of measures* $\{\mu_t \times \delta_f\}_{t\in(0,T)}$ *is the statistical solution of the heat equation corresponding to the given measures* $\mu_0$ *and* $\zeta = \delta_f$ *(in the sense of Definition* 3).

The proof is simple and natural. It is enough to compare Definitions 1–3 with Def. 1, 2, 4 of [2].

The statistical solution is a generalization of the weak solution. As an immidiate consequence of Theorem 2 and Theorem 5 from [2] we have

COROLLARY. *The following conditions are equivalent*:

(i) *the function* $u$: $(0, T) \to H_0^1$ *is the weak solution of the heat equation with initial value* $u_0$ *and right side term* $f$,

(ii) *the family of measures* $\{\delta_{u(t)} \times \delta_f\}_{t\in(0,T)}$ *is the statistical solution of the heat equation corresponding to* $\mu_0 = \delta_{u_0}$, $\zeta = \delta_f$.

**4. The existence theorem.** At the beginning we introduce the new class of test functionals.

Definition 4. The test functional $F$ will be called *elementary* if for some $n$ (depending on $F$) we have $F(u,f) = F(P_n u, f)$. $P_n$ denotes the orthogonal projection of $L^2$ on the subspace spanned by $w_1, \ldots, w_n$, where $\{w_i\}_{i=1,2,\ldots}$ is the orthonormal basis of $L^2$ built from eigenfunctions of the operator $A$.

LEMMA 1. *If the equality* (3) *holds for all elementary test functionals then it holds for all test functionals.*

Proof. We take any test functional $F$ and we put

$$F_n(u,f) := F(P_n u, f).$$

The functionals $F_n$ are elementary test functionals. (3) holds true for each $F_n$. Letting $n \to \infty$ we obtain (3) for $F$ (the detailed proof is similar to the proof of Lemma 1 in [2]).

Let us assume that $K_1 := \operatorname{supp}\mu_0$ and $K_2 := \operatorname{supp}\zeta$ are compact sets in $H_0^1$ and $L^2(0, T; L^2)$ respectively. The set $K_t := S(t, K_1, K_2)$ is compact in $H_0^1$. Let $\varphi$ be any continuous function defined on the set $K_t \times K_2$. The functional

$$\varphi \to \int_{K_1\times K_2} \varphi(S(t,u,f),f)\, d\mu_0 \times \zeta(u,f)$$

is linear and bounded. There exists the unique probability, Borel measure on the set $K_t \times K_2$ such that

$$\int_{K_t \times K_2} \varphi(u,f)\,d\nu_t(u,f) = \int_{K_1 \times K_2} \varphi(S(t,u,f),f)\,d\mu_0 \times \zeta(u,f) \tag{4}$$

It is not difficult to prove

LEMMA 2. *The family of measures* $\{\nu_t\}_{t \in (0,T)}$ *is basic.* In the proof we apply Theorem 1 p. 4) and the equality (4).

THEOREM 3. *The family of measures* $\{\nu_t\}_{t \in (0,T)}$ *is the statistical solution of the heat equation corresponding to* $\mu_0$ *and* $\zeta$.

Proof. It is enough to show that (3) holds for each elementary test functional $F$. The function

$$t \to F(S(t,u,f),f)$$

has the weak derivative and

$$\frac{\partial}{\partial t} F(S(t,u,f),f) = \frac{\partial}{\partial t} S(t,u,f)\,[F_1'(S(t,u,f),f)]$$

(the derivative $\frac{\partial}{\partial t} S(t,u,f)$ is a member of $H_0^{-1}$ in virtue of Theorem 1 p. 2)). The above fact has been proved in [2] (the beginning of the proof of Theorem 4). Applying (1) we get

$$\frac{\partial}{\partial t} F(S(t,u,f),f) = -\big(S(t,u,f),\, F_1'(S(t,u,f),f)\big)_1 + \big(f(t),\, F_1'(S(t,u,f),f)\big). \tag{5}$$

Now we shall prove that the function

$$t \to \int_M F(S(t,u,f),f)\,d\mu_0 \times \zeta(u,f)$$

has a weak derivative equal to

$$g(t) := \int_M \frac{\partial}{\partial t} F(S(t,u,f),f)\,d\mu_0 \times \zeta(u,f).$$

At first we observe that the function $g$ makes sense and is integrable on $(0,T)$. For almost all $t \in (0,T)$ we have

$$\begin{aligned}
\left|\frac{\partial}{\partial t} F(S(t,u,f),f)\right| &\leqslant \left\|\frac{\partial}{\partial t} S(t,u,f)\right\|_{-1} \cdot \|F_1'(S(t,u,f),f)\|_1 \\
&\leqslant c_9(\|A_e S(t,u,f)\|_{-1} + \|f(t)\|_{-1}) \\
&\leqslant c_9 \|S(t,u,f)\|_1 + c_1 c_9 \|f(t)\|.
\end{aligned}$$

The right side of this inequality is integrable on $(0, T)$. Hence in virtue of (2) we obtain

$$\int_0^T \left|\frac{\partial}{\partial t} F(S(t,u,f),f)\right| dt \leqslant c_{11}+c_{12}\|u\|^2+c_{13}\|f\|^2_{L^2(0,T,L^2)}$$

with suitable constants $c_{11-13}$. The right side of the last inequality is $\mu_0\times\zeta$ integrable so that

$$\int_M \int_0^T \left|\frac{\partial}{\partial t} F(S(t,u,f),f)\right| dt\, d\mu_0\times\zeta(u,f) < +\infty . \tag{6}$$

Thus thanks to Fubini's theorem $g$ is integrable. We take any $h\in C_0^1(0, T; R)$ and applying Fubini's theorem twice (in virtue of (6)) we get

$$-\int_0^T h'(t)\int_M F(S(t,u,f),f)\,d\mu_0\times\zeta(u,f)\,dt = \int_0^T h(t)g(t)\,dt .$$

Finally we have

$$\frac{d}{dt}\int_M F(S(t,u,f),f)\,d\mu_0\times\zeta(u,f) = \int_M \frac{\partial}{\partial t}F(S(t,u,f),f)\,d\mu_0\times\zeta(u,f)$$

for almost all $t\in(0, T)$. Substituting (5) we obtain

$$\frac{d}{dt}\int_M F(S(t,u,f),f)\,d\mu_0\times\zeta(u,f)+\int_M (S(t,u,f),\,F_1'(S(t,u,f),f))_1\,d\mu_0\times\zeta(u,f)$$
$$= \int_M (f(t),\,F_1'(S(t,u,f),f))\,d\mu_0\times\zeta(u,f) .$$

We can eliminate the function $S$ using the formula (4). Integrating the result from 0 to $s$ we finish the proof of the theorem.

**5. The uniqueness theorem.** We introduce the following notations:

$$B_r := \{(u,f)\in M\colon\ (\|u\|^2+\|f\|^2_{L^2(0,T;L^2)})^{1/2}\leqslant r\} ,$$
$$\mathscr{M}_r := \{G\colon B_r\to R\colon \text{ for a test functional } F\ G = F|_{B_r}\} ,$$

where $r$ is arbitrarily fixed in $(0, \infty)$.

LEMMA 3. *$\mathscr{M}_r$ is the dense set in the set of all weakly continuous functions defined on $B_r$.*

Proof. In virtue of the Stone–Weierstrass theorem it is sufffcient to prove that $\mathscr{M}_r$ is an algebra with 1, separating points of $B_r$. Let us suppose that for some $(u_1,f_1)\neq(u_2,f_2)\in B_r$ and all $G\in\mathscr{M}_r$ we have $G(u_1,f_1) = G(u_2,f_2)$. Putting $F_n(u,f) := (u, w_n)$, where $\{w_n\}$ is the basis of $L^2$, we obtain the sequence of test functionals. Hence $F_n|_{B_r} \in \mathcal{M}_r$ so that $(u_1, w_1) = (u_2, w_n)$ $(n = 1, 2, \ldots)$. This yields $u_1 = u_2$ and consequently $f_1 \neq f_2$. Now we take

$$F(u,f) := \int_a^b \|f(t)\| \, dt,$$

where the interval $[a, b] \subset (0, T)$ is chosen to get

$$\int_a^b \|f_1(t)\| \, dt \neq \int_a^b \|f_2(t)\| \, dt.$$

We take any continuous $H: L^2(0, T; L^2) \to [0, 1]$ such that

$$H(f) = \begin{cases} 0 & \text{if } \|f\|_{L^2(0,T;L^2)} \geqslant r+1, \\ 1 & \text{if } \|f\|_{L^2(0,T;L^2)} \leqslant r. \end{cases}$$

It is easy to see that $F \cdot H$ is the test functional. $F \cdot H_{B_r} \in \mathcal{M}_r$ which leads to a contradiction. Thus $\mathcal{M}_r$ separates the points of $B_r$. Obviously $1 \in \mathcal{M}_r$. The fact that $\mathcal{M}_r$ is an algebra can be verified as in [2] (Lemma 4 p. 129).

THEOREM 4. *We assume that the support of $\mu_0 \times \zeta$ is bounded in $L^2 \times L^2$. Families of measures $\{v_t'\}_{t \in (0,T)}$, $\{v_t''\}_{t \in (0,T)}$ are supposed to be defined on the space $L^2 \times L^2$ too. If they both are the statistical solution of the heat equation corresponding to the measures $\mu_0$ and $\zeta$, and if the supports of $v_t'$, $v_t''$ are uniformly bounded then $v_t' = v_t''$.*

Proof. After slight modifications we can repeat the proof of Theorem 7 of [2] getting

$$\int_{B_r} F(u,f) \, dv_t'(u,f) = \int_{B_r} F(u,f) \, dv_t''(u,f)$$

for all test functionals $F$. Now applying Lemma 3 we obtain the above equality for all weakly continuous functions. Hence the measures $v_t'$ and $v_t''$ are equal on "weakly" Borel sets in $L^2 \times L^2$ but each Borel set is also "weakly" Borel set so it completes the proof of the theorem.

Remarks. It is easy to see that the family of measures constructed in Section 4 is the unique statistical solution of the heat equation with uniformly bounded support corresponding to the given measures $\mu_0$, $\zeta$ with compact supports in $L^2$.

Upon surmounting some technical difficulties the existence theorem may be generalised to the case including the given measures with bounded support.

## References


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