

Asymptotic properties of discontinuous solutions of a functional equation

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We consider the linear homogeneous functional equation

$$(1) \quad \varphi(f(x)) = g(x)\varphi(x)$$

in an interval $I := (0, A]$ or $(0, A)$, where $0 < A \leq +\infty$. In the sequel we put $I^* := I \cup \{0\}$.

We impose the following hypotheses on the given function f and g :

(H) $f, g: I^* \rightarrow \mathbb{R}$ are continuous in I^* and $0 < f(x) < x, g(x) > 0$ in I . Moreover, f and g have the following asymptotic properties as $x \rightarrow 0+$:

$$f(x) = x - ax^{m+1} + o(x^{m+1+\mu})$$

$$g(x) = 1 + bx^k + o(x^{k+\varkappa}),$$

where $a, b, m, k, \mu, \varkappa$ are some positive constants.

We aim at determining the asymptotic behaviour (as $x \rightarrow 0+$) of solutions of (1) which are continuous in I but not necessarily in I^* . If not otherwise stated, in the sequel all the asymptotic symbols in this paper refer to $x \rightarrow 0+$.

Solutions of (1) which are continuous in I^* have been investigated in [2], cf. also [3], ch. II. It turns out that there are three possible cases:

(A) there is one parameter family of solutions continuous in I^* ,

(B) the solution depends on an arbitrary function, and

(C) $\varphi(x) \equiv 0$ is the only continuous solution of (1) in I^* , according as the sequence

$$(2) \quad G_n(x) := \prod_{i=0}^{n-1} g(f^i(x)), \quad n = 1, 2, \dots; \quad G_0(x) \equiv 1$$

converges in I^* to a continuous never vanishing function, or converges uniformly to zero on a subinterval of I , or else.

Under our hypotheses (H) the case actually occurring depends on the mutual relation between k and m :

LEMMA. Assume (H) to hold with the asymptotic relation for the function g replaced by the following one if $k > m$:

$$(3) \quad g(x) = 1 + o(x^k).$$

If $k > m$ then case (A) occurs, and if $k \leq m$ then (C) is the occurring case.

Proof. First we shall prove that for every $d \in I$ there are: an $M_d > 0$ and an $N_d = N(M_d, d) \in \mathbb{N}$ such that we have

$$(4) \quad f^n(x) \leq M_d n^{-1/m} \quad \text{for } n \geq N_d, x \in [0, d]$$

(here, as in (2), upper indices denote functional iterates).

To this end we define the function

$$F(x) := \sup \{f(t), t \in [0, x]\}$$

and prove that

$$(5) \quad F(x) = x - x^{m+1}u(x), \quad x \in I, \quad \liminf_{x \rightarrow 0^+} u(x) > 0.$$

Indeed, for every $x \in I$ there is a $z \in [f(x), x]$ to fulfil $f(z) = F(x)$. This is so since the function f , being continuous in $[0, x] \subset I^*$ takes on the value of its upper bound at a point, say $z \in [0, x]$; and $z < f(x) \leq F(x) = f(z)$ contradicts assumption (H). Write the asymptotic relation for f in the form similar to (5):

$$f(x) = x - x^{m+1}v(x), \quad \lim_{x \rightarrow 0^+} v(x) = a > 0.$$

Now we have

$$u(x) = (x - F(x))x^{-m-1} \geq (z - f(z))x^{-m-1} = (z/x)^{m+1}v(z) \geq (f(x)/x)^{m+1}v(z).$$

Passing to the limit as $x \rightarrow 0^+$ we get $\liminf_{x \rightarrow 0^+} u(x) \geq a > 0$, i.e. relation (5) holds true.

We know that the function F is increasing in I^* and satisfies (5). Thus, by a result from [4], for every $d \in I$ we can find an $M_d > 0$ and an $N_d \in \mathbb{N}$ such that $F^n(x) \leq F^n(d) \leq M_d n^{-1/m}$ for $n \geq N_d$ and every $x \in [0, d]$. By the inequality $f^n(x) \leq F^n(x)$, resulting from the definition and the monotonicity of the function F , we obtain inequality (4).

Now, let $k > m$. We shall show that the series

$$(6) \quad \sum_{n=0}^{\infty} (g(f^n(x)) - 1)$$

is absolutely and uniformly convergent in every interval $[0, d] \subset I^*$. For, from (3) we have the inequality

$$|g(x) - 1| \leq Kx^k, \quad x \in [0, c] \subset I^*$$

with some positive K and $0 < c \leq d$. Thus (by (4))

$$|g(f^n(x)) - 1| \leq K(f^n(x))^k \leq KM_d^k n^{-k/m}$$

for $n \geq N \geq N_d$ such that $f^N(d) \leq c$ and every $x \in [0, d]$. Since $k > m$, this shows that series (6) actually converges, absolutely and uniformly in every interval $[0, d] \subset I^*$. Thus the sequence (2) approaches a continuous and positive limit in I^* , i.e. case (A) occurs.

If $k \leq m$, then for every fixed $x \in I$ the asymptotic property of the function f implies the relation (cf. [4])

$$f^n(x) = n^{-1/m}((am)^{-1/m} + o(1)), \quad n \rightarrow \infty.$$

This, together with the asymptotic property of the function g (occurring in (H)), yields

$$|g(f^n(x)) - 1| > b_1(f^n(x))^k > B(x)n^{-k/m} > 0$$

(where $0 < b_1 < b$), at least from an index $N(x) \in \mathbb{N}$ onwards. Since the series $\sum n^{-k/m}$ now diverges, then so does series (6), for every $x \in I$. Thus $G_n(x) \rightarrow +\infty$ for every $x \in I$. This excludes both cases (A) and (B), i.e. for sequence (2) case (C) occurs, which was to be proved.

Remark. We have case (B) whenever $k \leq m$ and $b < 0$, cf. [1]. Moreover, the facts stated in the Lemma can be found in the same paper [1]. Proofs of them, however, are supplied under some more restrictive assumptions on the asymptotic behaviour of the function g , and in the case where f is strictly increasing in I^* .

Without assuming anything on the sign of b we have the following simple result.

THEOREM 1. *Under hypotheses (H) (with $b \neq 0$ of arbitrary sign) if equation (1) has a solution φ_0 which is continuous and positive in I then any other solution φ of (1) has the property:*

$$\varphi(x) = \varphi_0(x) \omega(x).$$

Proof. Put

$$\omega(x) := \varphi(x)/\varphi_0(x).$$

Then ω is a continuous solution of the equation

$$\omega(f(x)) = \omega(x),$$

thus, being continuous, is bounded in I .

The first author in paper [1] investigated the asymptotic behaviour of solutions of (1) which are continuous in I^* . In cases (A) and (B) equation (1) actually has solutions continuous in I^* which are positive in I . Their asymptotic behaviour is known from [1]. This, together with Theorem 1, settles the problem for cases (A) and (B). Therefore we restrict ourselves to case (C), i.e. where (cf. Lemma)

$$k \leq m.$$

First assume $k = m$. Then we may look for a particular solution of (1) with the asymptotic property $\varphi_0(x) \sim x^{-p}$, $p > 0$.

Putting

$$(7) \quad \psi(x) = x^p \varphi_0(x)$$

and inserting (7) into (1) we obtain that ψ must fulfill the functional equation

$$(8) \quad \psi(f(x)) = g_1(x)\psi(x),$$

where

$$g_1(x) = g(x)(f(x)/x)^p.$$

Under our hypotheses (H) we have

$$g_1(x) = (1 + bx^m + o(x^{m+\alpha}))(1 - ax^m + o(x^{m+\beta}))^p = 1 + (b - ap)x^m + o(x^{m+\nu}),$$

where $v = \min(\mu, \kappa, m) > 0$, so that $m+v > m$. Choosing $p = b/a$ we make the term containing x^m in g_1 vanish, so that by Lemma for equation (8) case (A) occurs.

Thus equation (8) has the one-parameter family of solutions which are continuous in I^* and do not vanish in I (except for the trivial one $\psi(x) \equiv 0$), and so they have the property $\psi(x) = 0(1)$. Consequently equation (1) has a positive solution φ_0 with the property

$$(9) \quad \varphi_0(x) = 0(x^{-b/a}),$$

and by Theorem 1 every solution of (1) continuous in I has property (9). Thus we have proved

THEOREM 2. *Let hypotheses (H) be fulfilled with $k = m$. Then every solution φ of equation (1) continuous in I has the asymptotic property (9).*

Now we pass to the case $k < m$. We look for a particular solution φ_0 of (1) with the asymptotic property

$$\varphi_0(x) \sim \exp(rx^{-p})$$

with some positive r and p .

Putting

$$(10) \quad \psi(x) = \varphi_0(x) \exp(-rx^{-p})$$

and inserting (10) into (1) we obtain that the function ψ must fulfil equation (8), but now

$$g_1(x) = g(x) \exp(-r[f(x)^{-p} - x^{-p}]).$$

By assumption (H) we have

$$f(x)^{-p} = x^{-p}(1 - ax^m + 0(x^{m+\mu}))^{-p} = x^{-p}(1 + pax^m + 0(x^{m+\delta})),$$

where $\delta = \min(\mu, m) > 0$. Hence

$$f(x)^{-p} - x^{-p} = x^{-p}(pax^m + 0(x^{m+\delta})).$$

Since $\exp u = 1 + u + 0(u^2)$ we get

$$\exp(-r[f(x)^{-p} - x^{-p}]) = 1 - rpa x^{m-p} + 0(x^{m-p+\gamma}),$$

where $\gamma = \min(\delta, m-p) = \min(\mu, m-p)$. Thus the optimal choice of p is $p = m-k$, so that $m-p = k$. Then

$$g_1(x) = (1 + bx^k + 0(x^{k+\kappa}))(1 - rpa x^k + 0(x^{k+\gamma})) = 1 + (b - rpa)x^k + 0(x^{k+\tau}),$$

where $\tau = \min(k, \kappa, \gamma) = \min(k, \kappa, \mu) > 0$.

In order to make the coefficient of x^k in g_1 vanish we choose $r = b/pa = b/a(m-k)$. Now the behaviour of solutions of equation (8) continuous in I^* depends on the term $0(x^{k+\tau})$. If $k+\tau > m$, we have case (A) (cf. Lemma) and the solutions of equation (1) have the asymptotic property

$$(11) \quad \varphi(x) = 0(\exp([b/a(m-k)]x^{k-m})).$$

Thus we have proved

THEOREM 3. Let hypotheses (H) be fulfilled with $k < m$. Let moreover $\min(k, \kappa, \mu) > m - k$. Then every solution φ of equation (1) continuous in I has the asymptotic property (11).

If $\min(k, \kappa, \mu) \leq m - k$ nothing can be said about the asymptotic behaviour of solutions of (1) without any further assumptions. If we have better information about the asymptotic behaviour of f and g , then we can write $g_1(x)$ in a form

$$g_1(x) = 1 + b_1 x^{k+\tau} + o(x^{k+\tau+\sigma}), \quad \sigma > 0$$

and repeat the preceding argument. After several steps we arrive at an asymptotic formula

$$\varphi(x) = o(\exp[-r_0 \log x + r_1 x^{-p_1} + \dots + r_n x^{-p_n}])$$

with all the constants r_i ($i = 0, \dots, n$) and p_j ($j = 1, \dots, n$) positive.

To illustrate these considerations we present the following example.

Example. Consider the equation

$$(12) \quad \varphi(x - x^3) = (1 + x + x^2)\varphi(x), \quad I = (0, 1).$$

Here $m = 2$, $\mu = +\infty$; $k = 1$, $\kappa = 1$; $a = b = 1$; whence $p = r = 1$. Putting $\psi(x) = \varphi(x)\exp(-1/x)$ we get for ψ equation (8) with

$$\begin{aligned} g_1(x) &= (1 + x + x^2)\exp(-x^{-1}(1/(1-x^2)-1)) \\ &= (1 + x + x^2)\exp(-x + o(x^3)) \\ &= (1 + x + x^2)(1 - x + x^2/2 + o(x^3)) \\ &= 1 + x^2/2 + o(x^3). \end{aligned}$$

We have for the function g_1 in (8): $k_1 = m = 2$, $b_1 = 1/2$, $\kappa_1 = 1$. Thus theorem 2 applies for our equation (8), yielding $\psi(x) = o(x^{-1/2})$ for solutions ψ of (12) continuous in I . Finally, all the solutions φ of equation (1) that are continuous in I have the asymptotic property $\varphi(x) = o(x^{-1/2}\exp(1/x))$.

References

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