

A note on the Hammerstein operator in Köthe spaces

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1. This Note is concerned with the Hammerstein nonlinear integral operator $Au(x) = \int_X k(x, y) \cdot f(y, u(y)) d\mu(y)$ defined on Köthe space L_ϱ , where ϱ has the Fatou property.

The class of Köthe spaces L_ϱ with ϱ satisfying the Fatou condition contains L^p ($1 \leq p \leq \infty$) spaces, Orlicz, Marcinkiewicz and Lorentz spaces.

We state sufficient conditions for Lipschitz property of the Hammerstein operator in L_ϱ .

2. **Normed Köthe spaces** (cf. [2] or [4]). Let (X, μ, A) be a measure space. For the sake of simplicity we will assume that the measure μ is σ -finite. By P we will denote the set of all non-negative μ -measurable functions, where we will permit also that a function in P assumes the value $+\infty$ on a set of positive measure.

We assume now that to each $f \in P$ there corresponds a number $\varrho(f)$ such that

- 1) $0 \leq \varrho(f) \leq \infty$,
- 2) $\varrho(f) = 0 \Leftrightarrow f(x) = 0$ for μ -almost every x ,
- 3) $\varrho(af) = a\varrho(f)$ for every finite constant $a \geq 0$,
- 4) if $f, g \in P$ and $f(x) \leq g(x)$ for μ -almost every x , then $\varrho(f) \leq \varrho(g)$,
- 5) $\varrho(f+g) \leq \varrho(f) + \varrho(g)$ for all $f, g \in P$.

The function ϱ thus defined on P and mapping P into the non-negative numbers is called a function norm.

The definition of ϱ is extended to complex valued μ -measurable f by defining that $\varrho(f) = \varrho(|f|)$.

We denote by L_ϱ the set of all f satisfying $\varrho(f) < \infty$. The function ϱ is a norm on L_ϱ ; the normed linear space L_ϱ is now called a normed Köthe space. The spaces L^p ($1 \leq p \leq \infty$), Orlicz spaces, Marcinkiewicz spaces, Lorentz spaces are examples of normed Köthe spaces.

The function norm ϱ is said to have the Riesz-Fischer property if, for any sequence f_n ($n = 1, 2, \dots$) in L_ϱ such that $\sum_1^\infty \varrho(f_n) < \infty$, we have $\sum_1^\infty |f_n| \in L_\varrho$, i.e. $\varrho(\sum_1^\infty |f_n|) < \infty$. It was proved that the space L_ϱ is a Banach space if and only if ϱ has the Riesz-Fischer (cf. [4]). The complete normed Köthe spaces are frequently called Banach function spaces.

The function norm ϱ is said to have the Fatou property if it follows from $0 \leq f_1 \leq f_2 \leq \dots \uparrow f$ with all $f_n \in P$, that $\varrho(f_n) \nearrow \varrho(f)$. It can be proved easily that the Fatou property implies the Riesz-Fischer property, but not conversely (cf. [4]).

At this point we introduce another function norm defined for all measurable $f \in P$ by

$$\varrho'(f) = \sup_{\varrho(g) \leq 1} \int fg d\mu, \quad g \in P.$$

The norm ϱ' is called the associate norm of ϱ , and the corresponding Banach function space $L_{\varrho'}$ is called the associate space of L_{ϱ} .

For function $f \in L_{\varrho}$ and $g \in L_{\varrho'}$, the following inequality

$$\left| \int fg d\mu \right| \leq \int |fg| d\mu \leq \varrho(f) \varrho'(g)$$

holds (cf. [4]). This inequality is called the Hölder inequality on account of its obvious analogy with the classical Hölder inequality in the L^p case.

For any μ -measurable subset E of X , we shall denote the characteristic function of E by χ_E . If E is μ -measurable subset of X such that $\varrho(\chi_F) = \infty$ for every subset F of E satisfying $\mu(F) > 0$, then E is called a ϱ -purely infinite subset. If there exists no ϱ -purely infinite subset, the norm ϱ is said to be saturated.

It is not difficult to prove the associate ϱ' of the saturated function norm ϱ is again a function norm. It is a much deeper fact that ϱ' is also saturated (cf. [4]), but once this has been established it follows immediately that all higher associates $\varrho^{(n)}$ ($n = 2, 3, \dots$) are saturated function norms, where $\varrho^{(n)}$ is the associate of $\varrho^{(n-1)}$.

It is easy again to prove that $\varrho^{(n+2)} = \varrho^{(n)}$ for $n \geq 1$, but it may happen that ϱ and $\varrho^{(2)}$ are not equal. The norms ϱ and $\varrho^{(2)}$ always satisfy $\varrho^{(2)} \leq \varrho$, and $\varrho^{(2)} = \varrho$ holds if and only if ϱ has the Fatou property (cf. [4]). This key result is due to W. A. J. Luxemburg ([1]) and independently to G. G. Lorentz.

3. For the Hammerstein operator.

$$Au(x) = \int_X k(x, y) f(y, u(y)) d\mu(y)$$

one assumes usually that f satisfies the so called Caratheodory conditions:

1) $f(x, \cdot): R \ni u \rightarrow f(x, u) \in R$ is continuous for almost every $x \in X$,

2) $f(\cdot, u): X \ni x \rightarrow f(x, u) \in R$ is measurable for all $u \in R$,

which guarantee the measurability of $X \ni x \rightarrow f(x, u(x))$ for any measurable function $u: X \rightarrow R$.

Let the function $X \ni y \rightarrow k(x, y) \in R$ be a measurable function for almost every $x \in X$, then $k(x, y) f(y, u(y))$ is measurable in y for almost every $x \in X$ and for any measurable function u , therefore the integral

$$Au(x) = \int_X k(x, y) f(y, u(y)) d\mu(y)$$

has a well defined value in R under the following convention (cf. e.g. Rockafellar, [3]): if neither the positive nor the negative part of the function $y \rightarrow k(x, y) f(y, u(y))$ is summable

(i.e. finitely), we set $Au(x) = +\infty$. In this situation the value Au of the Hammerstein operator is always well defined member of P .

THEOREM. Let ϱ be a function norm with the Fatou property, let A denote the Hammerstein operator

$$Au(x) = \int_X k(x, y) f(y, u(y)) d\mu(y).$$

If there exists a measurable function $g: X \rightarrow R$ such that

- (i) $|f(x, u_1) - f(x, u_2)| \leq g(x)|u_1 - u_2|$ for every $u_1, u_2 \in R$ and for μ -almost every x ,
- (ii) $\{X \ni y \rightarrow k(x, y)g(y)\} \in L_{\varrho'}$ for μ -almost every x ,
- (iii) $\varphi \in L_{\varrho'}$ where $\varphi(x) = \varrho'(k(x, \cdot)g(\cdot))$,

then $\varrho(Au - Av) \leq \varrho(\varphi) \cdot \varrho(u - v)$ for every $u, v \in L_{\varrho}$ i.e. the operator A satisfies the Lipschitz condition with constant $\varrho(\varphi)$

Proof. Let $u, v \in L_{\varrho}$ and $w \in L_{\varrho'}$. We have

$$\begin{aligned} \left| \int_X [Au(x) - Av(x)] w(x) d\mu(x) \right| &\leq \left| \int_X \left(\int_X k(x, y) [f(y, u(y)) - f(y, v(y))] d\mu(y) \right) \cdot w(x) d\mu(x) \right| \\ &\leq \int_X \left(\int_X |k(x, y) [f(y, u(y)) - f(y, v(y))]| d\mu(y) \right) |w(x)| d\mu(x) \end{aligned}$$

By (i) we obtain that

$$\left| \int_X [Au(x) - Av(x)] w(x) d\mu(x) \right| \leq \int_X \left(\int_X |k(x, y)| g(y) |u(y) - v(y)| d\mu(y) \right) |w(x)| d\mu(x)$$

Since $k(x, \cdot)g(\cdot) \in L_{\varrho'}$ by (ii) and $|u(\cdot) - v(\cdot)| \in L_{\varrho}$ because $u, v \in L_{\varrho}$ it follows by Hölder inequality and definition of φ in (iii), that

$$\begin{aligned} \int_X \left(\int_X |k(x, y)g(y)| |u(y) - v(y)| d\mu(y) \right) |w(x)| d\mu(x) &\leq \int_X \varrho'(k(x, \cdot)g(\cdot)) \varrho(u - v) |w(x)| d\mu(x) \\ &= \int_X \varphi(x) \varrho(u - v) |w(x)| d\mu(x) = \varrho(u - v) \int_X \varphi(x) |w(x)| d\mu(x). \end{aligned}$$

Finally

$$(*) \quad \left| \int_X [Au(x) - Av(x)] w(x) d\mu(x) \right| \leq \varrho(u - v) \int_X \varphi(x) |w(x)| d\mu(x).$$

It follows from definition of $\varrho^{(2)}$ that

$$\varrho^{(2)}(Au - Av) = \sup_{\varrho'(w) \leq 1} \int_X [Au(x) - Av(x)] w(x) d\mu(x).$$

By (*) we have then

$$\begin{aligned} \varrho^{(2)}(Au - Av) &= \sup_{\varrho'(w) \leq 1} \int_X [Au(x) - Av(x)] w(x) d\mu(x) \\ &\leq \sup_{\varrho'(w) \leq 1} \varrho(u - v) \int_X \varphi(x) |w(x)| d\mu(x) \\ &\leq \varrho(u - v) \sup_{\varrho'(w) \leq 1} \int_X \varphi(x) |w(x)| d\mu(x) \\ &= \varrho(u - v) \varrho^{(2)}(\varphi). \end{aligned}$$

Finally $\varrho^{(2)}(Au - Av) \leq \varrho^{(2)}(\varphi)\varrho(u - v)$. Since ϱ has the Fatou property, it follows that $\varrho^{(2)} = \varrho$ (cf. remark in paragraph 2). We get then $\varrho(Au - Av) \leq \varrho(\varphi) \cdot \varrho(u - v)$, which is the desired result.

COROLLARY. *Under the assumptions of the previous Theorem the fact $\varrho(\varphi) < 1$ implies, by Banach fixed point theorem, that the equation*

$$\int_X k(x, y(f(y, u(y)))d\mu(y) = u(x)$$

has the unique solution in L_ϱ . Similarly we observe that the equation

$$\lambda \int_X k(x, y)f(y, u(y))d\mu(y) = u(x), \quad \lambda \neq 0$$

has the unique solution in L_ϱ for $|\lambda| < \frac{1}{\varrho(\varphi)}$.

References

- [1] W. A. J. Luxemburg, *Banach function spaces thesis* Delft Institute of Technology, Assen 1955.
- [2] W. A. J. Luxemburg, A. C. Zaanen, *Notes on Banach function spaces*, Konigl. Proc. Acad. Sci. Amsterdam 66 (2), (1963) 135-153.
- [3] R. T. Rockafellar, *Integral functionals, normal integrands and measurable selections*, Lecture Notes in Mathematics 543, Springer-Verlag, Berlin, Heidelberg, New York 1976.
- [4] A. C. Zaanen, *Integration*, Amsterdam 1967.

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